# THE SCOTS COLLEGE

# YEAR 12 HSC EXAMINATION

# EXTENSION 2 MATHEMATICS

# AUGUST 2007

**WEIGHTING:** **40%**

**TIME ALLOWED:** **3 HOURS**
*[plus 5 minutes reading time]*

**INSTRUCTIONS:**

- START EACH QUESTION IN A NEW BOOKLET.
- ALL QUESTIONS ARE OF EQUAL VALUE.
- ALL NECESSARY WORKING MUST BE SHOWN.
- BOARD APPROVED CALCULATORS MAY BE USED.
- DIAGRAMS ARE NOT TO SCALE.

## Question 1

**a.** Find $\int_0^{\frac{\pi}{2}} \sin^n x \cos x \, dx$ in simplest terms. **[3]**

**b.** You may assume that $\dfrac{1}{(2x+1)(x+2)} = \dfrac{2}{3(2x+1)} - \dfrac{1}{3(x+2)}$

**(i)** Use the method of partial fractions to show that $\int_0^1 \dfrac{dx}{(2x+1)(x+2)} = \dfrac{\ln 2}{3}$ **[2]**

**(ii)** Hence, evaluate $\int_0^{\frac{\pi}{2}} \dfrac{3\,dx}{4+5\sin x}$ using the substitution $t = \tan\left(\dfrac{x}{2}\right)$. **[3]**

**c.** Let $I_n = \int_0^1 x^n e^x \, dx$

**(i)** Evaluate $I_0$. **[1]**

**(ii)** Show $I_n = e - nI_{n-1}$ for $n \geq 1$. **[3]**

**(iii)** Hence evaluate $I_3 = \int_0^1 x^3 e^x \, dx$ **[3]**

## Question 2 **Start a new booklet**

**a.** Given $z = \sqrt{3} - i$ express:

**(i)** $z$ in the form $r \operatorname{cis} \theta$ **[1]**

**(ii)** $z^8$ in the form $a + ib$ **[2]**

**b.** $z$ is the complex number $x + iy$ and $|z-2| + |z+2| = 5$.

**(i)** Describe the locus of $z$ geometrically. **[3]**

**(ii)** Find the maximum and minimum values of $|z|$. **[1]**

**c.** Given $z = \cos\theta + i\sin\theta$ prove that:

**(i)** $z^n + \dfrac{1}{z^n} = 2\cos n\theta$ **[2]**

**(ii)** Express $x^5 - 1$ as the product of three factors each containing terms with real coefficients. **[4]**

**(iii)** Prove that $\left(1 - \cos\dfrac{2\pi}{5}\right)\left(1 - \cos\dfrac{4\pi}{5}\right) = \dfrac{5}{4}$ **[2]**

**QUESTION 3** **START A NEW BOOKLET**

**a.** **(i)** Sketch $f(x)=\ln(x-2)$ showing any intercepts and asymptotes. Now sketch on separate diagrams: **[1]**

**(ii)** $y=f(|x|)$ **[1]**

**(iii)** $|y|=f(x)$ **[1]**

**(iv)** $y^2=f(x)$ **[2]**

**(v)** $y=\dfrac{1}{f(|x|)}$ **[2]**

**b.** Sketch the function defined by: **[3]**

$f(x)=e^x, \quad 0\le x<1$

$f(x)=f(x+1), \quad 0\le x\le 4$

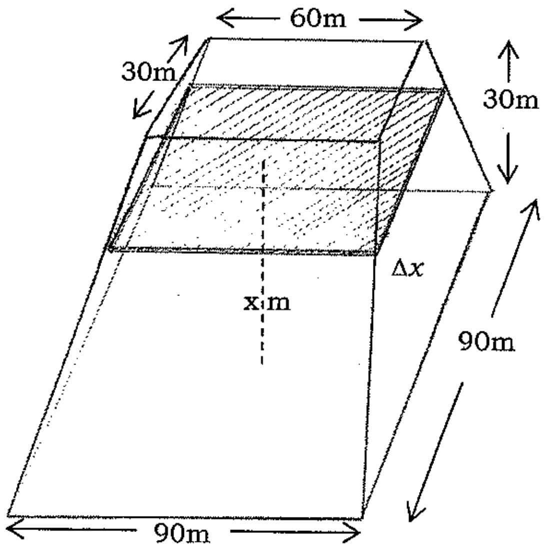
**c.** By considering domain and range and by finding any intercepts and asymptotes, sketch the graph of $y=\tan^{-1}\left(e^{-x}\right)$. **[5]**

**START A NEW BOOKLET**

**QUESTION 4**

**a.** Given $\int_0^k \sqrt{k^2 - x^2}\, dx = \frac{\pi}{2}$ find $k$. **[2]**

**b.** A burial chamber is located within a mound bounded on all sides by plane surfaces. Standing on a square base, side 90 metres, the mound is 30m high and its sides taper uniformly to a rectangular horizontal platform measuring 30m x 60m.




**(i)** Show the volume of a typical horizontal slice of thickness $\Delta x$ and lying $x$ metres above the ground is given by $\Delta V = 2(90-x)(45-x)\Delta x$ **[4]**

**(ii)** Find the volume of the burial mound. **[2]**

**c.** The arc defined by $y = e^x$, $0 \le x \le 1$ is rotated about the $x$ axis to form a curved bowl.

**(i)** Show the volume $V$ of the bowl, using the method of cylindrical shells is given by $V = \pi e^2 - 2\pi \int_1^e x \ln x\, dx$ **[3]**

**(ii)** Find the volume leaving your answer in exact form. **[3]**

**(iii)** Use the result in **(ii)** to evaluate $\int_0^1 e^{2x} dx$. **[1]**

**QUESTION 5** **START A NEW BOOKLET**

**a.** Express $\frac{2}{(1-x)(1+x^2)}$ in the form of the sum of two fractions with denominators $(1-x)$ and $(1+x^2)$. **[2]**

**b.** The equation $x^3 - ax + b = 0$ has roots $\alpha, \beta$ and $\delta$. Find the equation whose roots are $\frac{1}{2}\alpha$, $\frac{1}{2}\beta$ and $\frac{1}{2}\delta$. **[2]**

**c.** Find the equation of degree ten whose roots are the reciprocals of the roots of the equation $x^{10} - 5x^3 + x - 4 = 0$. **[2]**

**d.** Given that $x^5 - ax^3 + b = 0$ has a multiple root, show that $108a^5 - 3125b^2 = 0$. **[3]**

**e.** $P(x) = x^4 + 2x^3 + 9x^2 + 8x + 20$ has a zero $x = 2i - 1$.

**(i)** Evaluate $P(\overline{2i-1})$. **[1]**

**(ii)** Find the remaining zeros of $P(x)$. **[3]**

**(iii)** Factorise $P(x)$ over the real field. **[1]**

**(iv)** Factorise $P(x)$ over the complex field. **[1]**

**START A NEW BOOKLET**

**QUESTION 6**

**a.** A car travels around a banked circular track of radius 90 metres at 54km/hour.

**(i)** Draw a diagram showing all the forces acting on the car. **[1]**

**(ii)** Show that the car will have no tendency to slip sideways if the angle at which the track is banked is $\tan^{-1}\left(\frac{1}{4}\right)$. **[3]**

**(iii)** A second car of mass 1.2 tonnes travels around the same bend at 72km/hour. Find the sideways frictional force exerted by the road on the wheels of the car in Newtons. You may assume g = 10m/s$^2$. **[3]**

**b. (i)** The acceleration due to gravity at a point outside the earth is inversely proportional to the square of the distance $x$ from the centre of the earth. Neglecting air resistance, show that if a body is projected vertically upwards from the earth's surface, its speed $v$ ms$^{-1}$ in any position $x$ is given by $v^2 = u^2 - 2gR^2\left(\frac{1}{R} - \frac{1}{x}\right)$, where $R$ is the radius of the earth, $u$ ms$^{-1}$ is the initial speed and $g$ the acceleration due to gravity at the surface of the earth. **[4]**

**(ii)** Show that the greatest height $H$ above the surface of the earth achieved by the body is given by $H = \frac{u^2R}{2gR - u^2}$. **[2]**

**(iii)** Find the condition which will ensure the body escapes from the earth's influence. **[2]**

**START A NEW BOOKLET**

## QUESTION 7

**a.** The auxiliary circle $C: x^2 + y^2 = 4$ is supplied on the sheet attached to this paper. Place it in the appropriate booklet after entering your number.

**(i)** Use $C$ to construct the ellipse $x^2 + 4y^2 = 4$. **[2]**

**(ii)** Find the coordinates of $S^1$ and $S$ the focii of the ellipse. **[2]**

**(iii)** Find the equation of the tangent at the point on the ellipse where $x = \sqrt{3}$ and $y > 0$. **[2]**

**(iv)** Given $P$ is any point on the ellipse, find the perimeter of the triangle $S^1PS$. **[1]**

**b.** The point $P(a\sec\theta, b\tan\theta)$ lies on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ whose centre is O and focii $S^1$ and $S$.

**(i)** Show that $SP = a(e\sec\theta - 1)$. You may assume $S^1P = a(e\sec\theta + 1)$. **[3]**

**(ii)** Perpendiculars are drawn from $S^1$ and $S$ to meet the tangent at $P$ at $M$ and $N$ respectively.

Prove that $\sin\angle SPN = \sin\angle S^1PM$ and deduce that the tangent at $P$ bisects the angle $S^1PS$. **[5]**

**NOTE:** You may assume the equation of the tangent at P is

$$\frac{(\sec\theta)x}{a} - \frac{(\tan\theta)y}{b} = 1$$

**START A NEW BOOKLET**

## QUESTION 8

**a.** **(i)** Show that $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$ where $a$ is a constant. **[2]**

**(ii)** If $f(x)+f(p-x)=f(p)$ where $p$ is a constant, show that
$\int_0^p f(x)\,dx = \frac{1}{2}pf(p)$ **[2]**

**(iii)** Use the result of part **(i)** above to show that $\int_0^\pi \frac{x}{4+\sin^2 x}\,dx = \frac{\pi^2}{4\sqrt{5}}$ **[5]**

**b.** The angles of a triangle are such that the largest is one right angle in excess of the smallest. Given that the lengths of the sides of the triangle form an arithmetic sequence, find the ratios of the sides. **[6]**

SOLUTIONS EXT 2. 2007 TRIAL

Q1.(a) $\int_0^{\frac{\pi}{2}} \sin^n x \cos x \, dx$ Let $u = \sin x$ For $x = 0$, $u = 0$

$\frac{du}{dx} = \cos x$ For $x = \frac{\pi}{2}$, $u = 1$

$du = \cos x \, dx$

$= \int_0^1 u^n \, du$

$= \left[\frac{1}{n+1} u^{n+1}\right]_0^1$

$= \frac{1}{n+1}\left[(1) - (0)\right] = \frac{1}{n+1}$.

(b) (i) $\int_0^1 \frac{dx}{(2x+1)(x+2)} = \int_0^1 \frac{1}{3}\cdot\frac{2}{2x+1} - \frac{1}{3}\cdot\frac{1}{x+2} \, dx$

$= \frac{1}{3}\left[\ln(2x+1) - \ln(x+2)\right]_0^1$

$= \frac{1}{3}\left[\ln \frac{2x+1}{x+2}\right]_0^1$

$= \frac{1}{3}\left[\ln 1 - \ln \frac{1}{2}\right] = \frac{1}{3}\ln 2$.

(ii) $\int_0^{\frac{\pi}{2}} \frac{3 \, dx}{4+5\sin x} = \int_0^1 \frac{3 \cdot \frac{2\,dt}{1+t^2}}{4+5\cdot\frac{2t}{1+t^2}}$ For $t = \tan\frac{x}{2}$

$x = 0$, $t = 0$

$x = \frac{\pi}{2}$, $t = 1$

$= \int_0^1 \frac{6\,dt}{4(1+t^2)+10t}$

$= \int_0^1 \frac{3\,dt}{2(1+t^2)+5t}$

$= \int_0^1 \frac{3\,dt}{2t^2+5t+2}$

$= 3\int_0^1 \frac{dt}{(2t+1)(t+2)}$

$= 3\cdot\frac{1}{3}\ln 2$

$= \ln 2$.

SOLUTIONS EXT 2. SCOTS 2007.

Q1. (c)(i) $I_0 = \int_0^1 x^0 e^x\,dx = \int_0^1 e^x\,dx = [e^x]_0^1 = e-1$. ✓

(ii) $I_n = \int_0^1 x^n \frac{de^x}{dx}\,dx = [x^n e^x]_0^1 - \int_0^1 n x^{n-1} e^x\,dx$ ✓

so $I_n = [e-0] - n\int_0^1 x^{n-1} e^x\,dx$ ✓

$I_n = e - nI_{n-1}$. ✓

(iii) $I_3 = e - 3I_2$ ✓

$= e - 3[e - 2I_1]$

$= e - 3[e - 2(e - I_0)]$ ✓

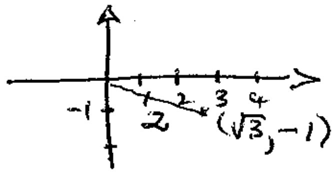
$= e - 3e + 6(e - (e-1))$

$= -2e + 6e - 6e + 6$

$= 6 - 2e$. ✓

Q2. (a) (i) $z = \sqrt{3} - i$

$z = 2\,\text{cis}\left(-\frac{\pi}{6}\right)$




(ii) $z^8 = 2^8\left(\text{cis}\left(-\frac{\pi}{6}\right)\right)^8$

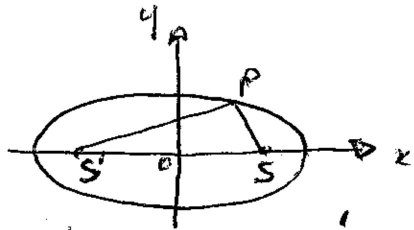
$= 256\,\text{cis}\left(-\frac{4\pi}{3}\right)$

$= 256\,\text{cis}\left(\frac{2\pi}{3}\right)$

$= 256\left(-\frac{1}{2} + \frac{\sqrt{3}}{2} i\right)$

$= -128 + 128\sqrt{3}\, i$

(b) (i) $|z-2| + |z+2| = 5$




SINCE $2a = 5$

$a = \frac{5}{2}$

NOTE: $PS + PS' = 2a$

A PARABOLA WITH FOCII $S(2,0)$ AND $S'(-2,0)$.

SO $ae = 2$

$e = \frac{2}{\frac{5}{2}}$

$e = \frac{4}{5}$

NOW $a^2e^2 = a^2 - b^2$

$b^2 = a^2 - a^2e^2$

$= a^2(1-e^2)$

$b^2 = \frac{25}{4}\left(1 - \frac{16}{25}\right)$

$b^2 = \frac{9}{4}$

AND $b = \frac{3}{2}$

A PARABOLA, FOCII $(2,0)$ AND $(-2,0)$
SEMI MAJOR AXIS $\frac{5}{2}$ UNITS
SEMI MINOR AXIS $\frac{3}{2}$ UNITS. $e = \frac{4}{5}$

(ii) MAXIMUM VALUE OF $|z| = \frac{5}{2}$

MINIMUM $|z| = \frac{3}{2}$

Q(2) (c) (i) $z^n + \frac{1}{z^n} = (\cos\theta + i\sin\theta)^n + (\cos\theta + i\sin\theta)^{-n}$

$= \cos n\theta + i\sin n\theta + \cos(-n\theta) + i\sin(-n\theta)$

$= \cos n\theta + i\sin n\theta + \cos n\theta - i\sin n\theta$

$= 2\cos n\theta$.

(ii) Consider $z^5 = 1$ Let $z = \cos\theta + i\sin\theta$

$(\cos\theta + i\sin\theta)^5 = 1$

$\cos 5\theta + i\sin 5\theta = 1$ and equating real parts

$\cos 5\theta = 1$.

$5\theta = 0, 2\pi, 4\pi, 6\pi$ and $8\pi$

$\theta = 0, \frac{2\pi}{5}, \frac{4\pi}{5}, \frac{6\pi}{5}, \frac{8\pi}{5}$

The roots of $z^5 = 1$ are: $\text{cis}\, 0, \text{cis}\frac{2\pi}{5}, \text{cis}\frac{4\pi}{5}, \text{cis}\frac{6\pi}{5}, \text{cis}\frac{8\pi}{5}$

The factors of $z^5-1$ are: $(z-\text{cis}\,0)(z-\text{cis}\frac{2\pi}{5})(z-\text{cis}\frac{8\pi}{5})(z-\text{cis}\frac{4\pi}{5})(z-\text{cis}\frac{6\pi}{5})$

So $z^5 - 1 = (z-1)(z-\text{cis}\frac{2\pi}{5})(z-\text{cis}(-\frac{2\pi}{5}))(z-\text{cis}\frac{4\pi}{5})(z-\text{cis}(-\frac{4\pi}{5}))$

$z^5 - 1 = (z-1)(z^2 - 2\cos\frac{2\pi}{5}z + 1)(z^2 - 2\cos\frac{4\pi}{5} + 1)$

(iii) Now $z^5 - 1 = (z-1)(z^4 + z^3 + z^2 + z + 1)$

So $z^4 + z^3 + z^2 + z + 1 = (z^2 - 2\cos\frac{2\pi}{5}z + 1)(z^2 - 2\cos\frac{4\pi}{5} + 1)$

For $z = 1$, $1+1+1+1+1 = (1 - 2\cos\frac{2\pi}{5} + 1)(1 - 2\cos\frac{4\pi}{5} + 1)$

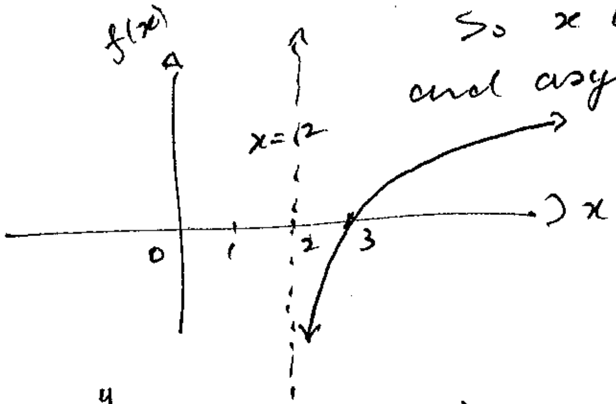
$5 = 2(1-\cos\frac{2\pi}{5}).2(1-\cos\frac{4\pi}{5})$

and $\frac{5}{4} = (1-\cos\frac{2\pi}{5})(1-\cos\frac{4\pi}{5})$ as required

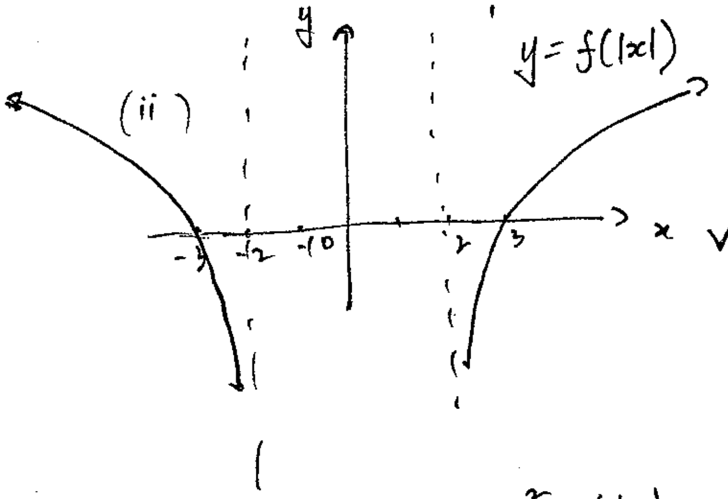
Q3.(a) $f(x) = \ln(x-2)$

$x - 2 = 1$ where $x = 3$.
$x - 2 = 0$ where $x = 2$

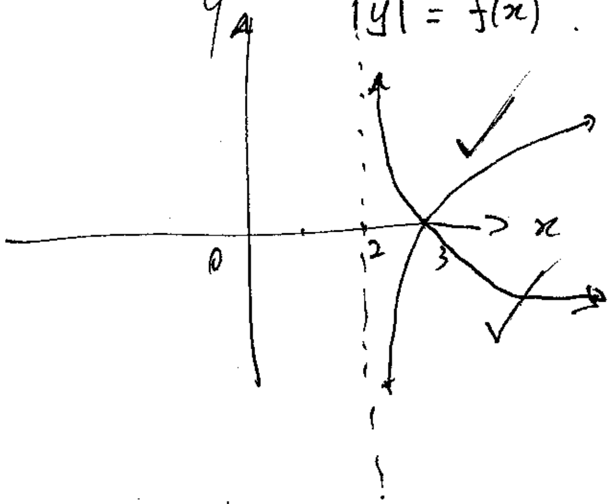
So $x$ intercept is $(3, 0)$
and asymptote is $x = 2$.




(ii) $y = f(|x|)$

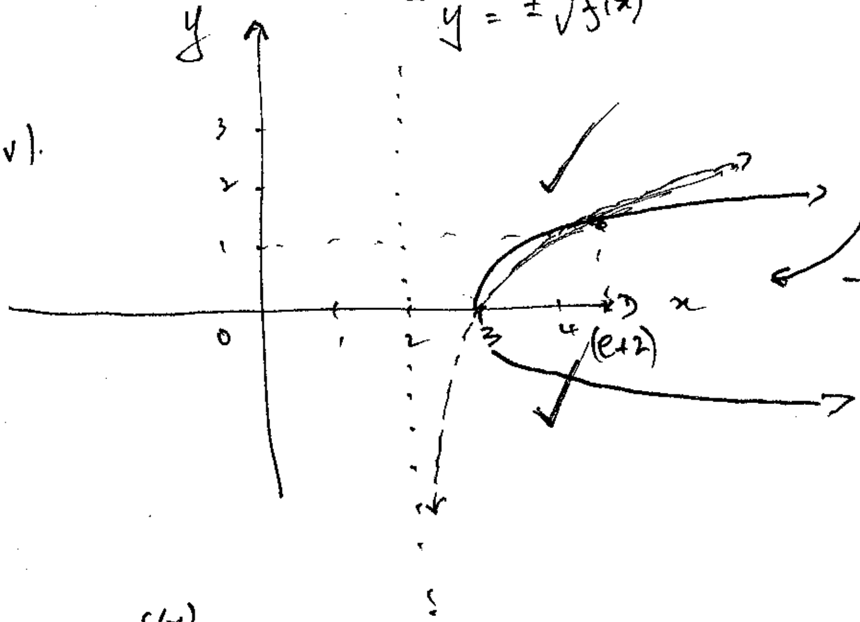


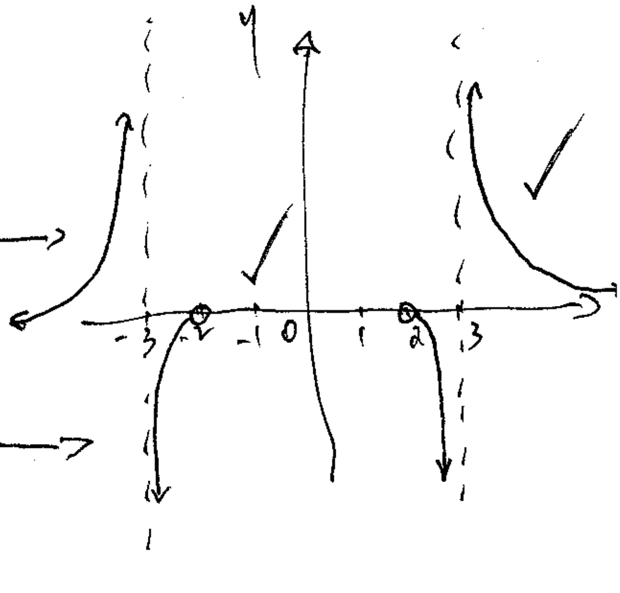

(iii) $|y| = f(x)$.



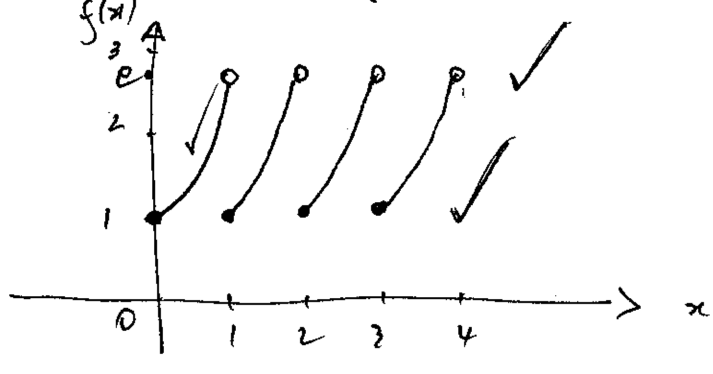

(iv).

$y^2 = f(x)$
$y = \pm\sqrt{f(x)}$




(b).



Domain: $x$ : $x$ All Reals ✓ Range: $y$ : $0 < y < \frac{\pi}{2}$ ✓

Q 3.

As $x \to -\infty$, $e^{-x} \to \infty$ and $\tan^{-1}(e^{-x}) \to \frac{\pi}{2}^-$

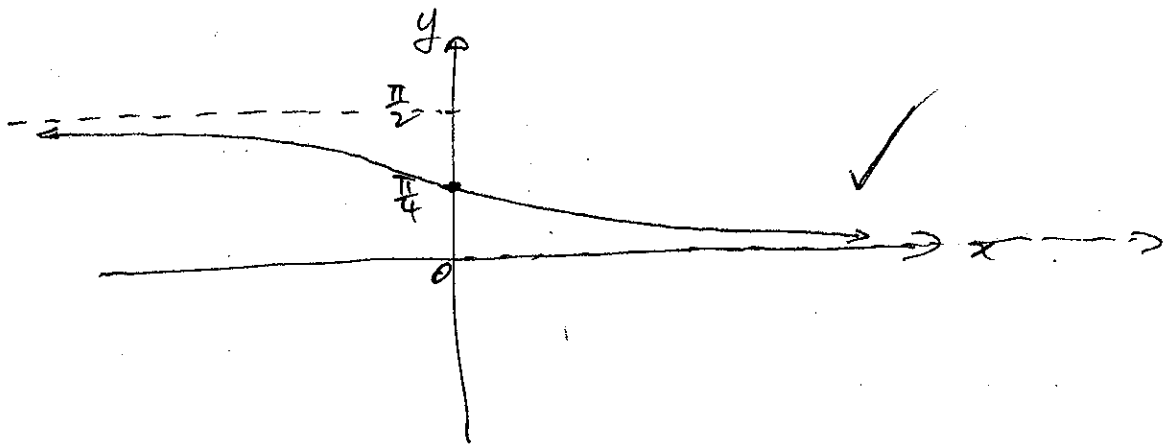
So $y = \frac{\pi}{2}$ is an asymptote as $x \to -\infty$

For $x = 0$, $e^{-x} = 1$ and $\tan^{-1}(e^{-x}) = \tan^{-1}(1) = \frac{\pi}{4}$

As $x \to \infty$, $e^{-x} \to 0^+$ and $\tan^{-1}(e^{-x}) \to 0^+$

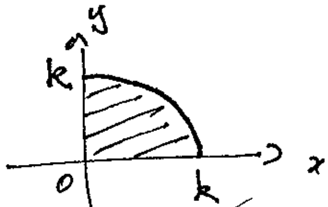
So $y = 0$ is an asymptote as $x \to \infty$

✓ ✓




✓

Q.4 (a) $I = \int_0^k \sqrt{k^2 - x^2} = \frac{\pi}{2}$



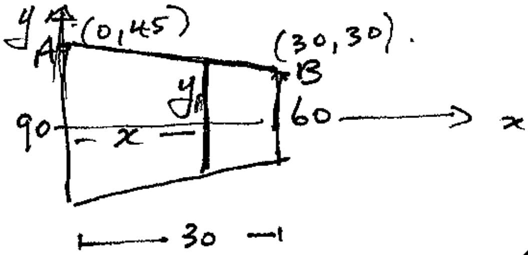

The integral represents $\frac{1}{4}$ of the area of a circle radius $k$. ✓

So $I = \frac{1}{4}\pi k^2$

So $\frac{1}{4}\pi k^2 = \frac{\pi}{2}$ ✓

$k^2 = 2$ and $k = \sqrt{2}$.

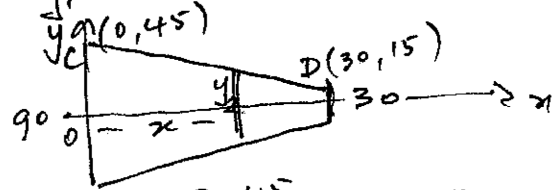
(b). From the front.




(i) Equation AB is $y_1 = \frac{30-45}{30-0} x + 45$ ✓

$y_1 = -\frac{1}{2}x + 45$ So $\boxed{2y_1 = 90 - x}$ ✓

From the side




Equat. CD is $y_2 = \frac{15-45}{30-0} x + 45$ ✓

$y_2 = -x + 45$ So $\boxed{2y_2 = 90 - 2x}$

Now $\Delta V = (2y_1)(2y_2).\Delta x$

$= 4(90-x)(90-2x)\Delta x$ ✓

$= 8(90-x)(45-x)\Delta x$.

(ii) Now $V = 8\int_0^{30} (90-x)(45-x)\, dx$

$= 8\int_0^{30} 4050 - 135x + x^2 \, dx$ ✓

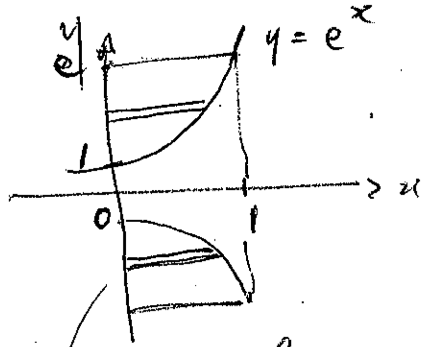
$= 8\left[4050x - \frac{135}{2}x^2 + \frac{1}{3}x^3\right]_0^{30}$

$= 8\{4050 \times 30 - \frac{135}{2} \times 900 + \frac{1}{3} \times 27000\}$

$= 8\{121500 - 60750 + 9000\}$ ✓

$= 139500 \text{ m}^3$

4(c) $y = e^x$




$x = \ln y$

By shells $V = \pi(e)^2 \times 1 - 2\pi\int_1^e (\ln y)\, y\, dy$

(i) $= \pi e^2 - 2\pi\int_1^e x\ln x\, dx$

(ii) $V = \pi e^2 - 2\pi\left[\int_1^e \frac{d(\frac{1}{2}x^2)}{dx}\ln x\, dx\right]$

$= \pi e^2 - 2\pi\left[\left[\frac{1}{2}x^2\ln x\right]_1^e - \int_1^e \frac{1}{2}x^2\frac{d(\ln x)}{dx}dx\right]$

$= \pi e^2 - 2\pi\left[\left[\left(\frac{1}{2}e^2\ln e\right) - \left(\frac{1}{2}\ln 1\right)\right] - \int_1^e \frac{1}{2}x\, dx\right]$

$= \pi e^2 - 2\pi\left[\frac{1}{2}e^2 - \left[\frac{1}{4}x^2\right]_1^e\right]$

$= \pi e^2 - 2\pi\left[\frac{1}{2}e^2 - \left[\frac{1}{4}e^2 - \frac{1}{4}\right]\right]$

$= \pi e^2 - \pi e^2 + 2\pi\left(\frac{1}{4}e^2 - \frac{1}{4}\right)$

$= \frac{\pi e^2}{2} - \frac{\pi}{2}$

$= \frac{\pi}{2}(e^2 - 1)$

(iii) By slices $V = \pi\int_0^1 (e^x)^2\, dx$

$= \pi\int_0^1 e^{2x}\, dx$

So $\pi\int_0^1 e^{2x}\, dx = \frac{\pi}{2}(e^2 - 1)$

and $\int_0^1 e^{2x}\, dx = \frac{1}{2}(e^2 - 1)$

Q5 (a) $\dfrac{2}{(1-x)(1+x^2)} = \dfrac{A}{1-x} + \dfrac{Bx+C}{1+x^2}$

So $2 \equiv A + Ax^2 + Bx + C - Bx^2 - Cx$

$2 = (A-B)x^2 + (B-C)x + A + C$

$A - B = 0$ — (1)

$B - C = 0$ — (2)

(1) + (2) $A - C = 0$ — (3)

and $A + C = 2$

so $2A = 2 \longrightarrow \boxed{A = 1}$, $\boxed{B = 1}$, $\boxed{C = 1}$

So $\dfrac{2}{(1-x)(1+x^2)} = \dfrac{1}{1-x} + \dfrac{x+1}{1+x^2}$

(b) Since $x = \alpha$ satisfies $\alpha^3 - a\alpha + b = 0$

and $8\left(\tfrac{1}{2}\alpha\right)^3 - 2a\left(\tfrac{1}{2}\alpha\right) + b = 0$

So $x = \tfrac{1}{2}\alpha$ is a root of $8x^3 - 2ax + b = 0$

(c) Let $\alpha$ be a root of $x^{10} - 5x^3 + x - 4 = 0$

so $\alpha^{10} - 5\alpha^3 + \alpha - 4 = 0$

and $1 - 5\left(\tfrac{1}{\alpha}\right)^7 + \left(\tfrac{1}{\alpha}\right)^9 - 4\left(\tfrac{1}{\alpha}\right)^{10} = 0$

So $x = \frac{1}{\alpha}$ is a root of $1 - 5x^7 + x^9 - 4x^{10} = 0$

or $4x^{10} - x^9 + 5x^7 - 1 = 0$

(d) So $5x^4 - 3ax^2 = 0$ has a root which is the multiple of $x^5 - ax^3 + b = 0$.

Now $5x^4 - 3ax^2 = 0$

or $x^2(5x^2 - 3a) = 0$

$x = 0$ or $x = \sqrt{\frac{3a}{5}}$. Since $x = 0$ is <u>not</u> a root of $x^5 - ax^3 + b = 0$, $x = \sqrt{\frac{3a}{5}}$ is the multiple root.

Substituting $\left(\sqrt{\frac{3a}{5}}\right)^5 - a\left(\sqrt{\frac{3a}{5}}\right)^3 + b = 0$

$\dfrac{9\sqrt{3}\,a^{\frac{5}{2}}}{25\sqrt{5}} - \dfrac{3\sqrt{3}\,a^{\frac{5}{2}}}{5\sqrt{5}} + b = 0$

$\dfrac{-6\sqrt{3}\,a^{\frac{5}{2}}}{25\sqrt{5}} = -b$

So $\dfrac{108a^5}{3125} = b^2$

$108a^5 - 3125b^2 = 0$

Q5 (e) If $x = -1+2i$ is a zero then
$x = -1-2i$ is a zero.

So $(x-(-1+2i))(x-(-1-2i))$ is a factor

$= (x+(1-2i))(x+(1+2i))$

$= x^2+2x+5$ a quadratic factor.

(i) $P(\overline{2i-1}) = 0$

(ii)

$$\begin{array}{r} x^2+4 \\ x^2+2x+5 \,\big)\, \overline{x^4+2x^3+9x^2+8x+20} \\ \underline{x^4+2x^3+5x^2} \\ 4x^2+8x+20 \\ \underline{4x^2+8x+20} \\ 0 \end{array}$$

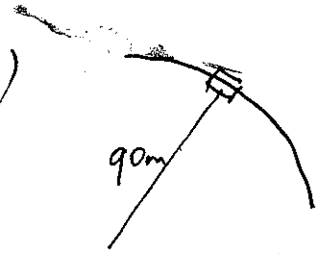
$P(x) = (x^2+2x+5)(x^2+4)$

Remaining zeros are $\pm 2i$

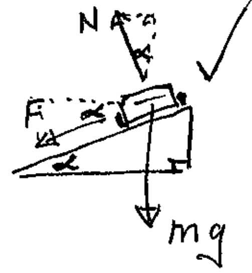
(iii) $P(x) = (x^2+2x+5)(x^2+4)$

(iv) $P(x) = (x+2i)(x-2i)(x+1-2i)(x+1+2i)$.

Q6. (a)




(i)




N - Normal reaction
F - Frictional force
mg - Weight

(ii) Resolving vertically

$$N\cos\alpha = F\sin\alpha + mg$$

$$N\cos\alpha - F\sin\alpha = mg \quad -(1)$$

horizontally $$N\sin\alpha + F\cos\alpha = m\frac{v^2}{r} \quad -(2)$$

From (1) and (2)

$$N\cos\alpha\sin\alpha - F\sin^2\alpha = mg\sin\alpha \quad -(3)$$

$$N\sin\alpha\cos\alpha + F\cos^2\alpha = m\frac{v^2}{r}\cos\alpha \quad -(4)$$

Subtract (4) - (3)

$$F(\cos^2\alpha + \sin^2\alpha) = m\frac{v^2}{r}\cos\alpha - mg\sin\alpha$$

$$F = m\frac{v^2}{r}\cos\alpha - mg\sin\alpha \quad *$$

No sideways slip means $F = 0$

and so $$mg\sin\alpha = m\frac{v^2}{r}\cos\alpha$$

$$\tan\alpha = \frac{v^2}{rg}$$

For $v = \frac{\cancel{54}^{3} \times \cancel{1000}^{5}}{\cancel{60}_{3} \times \cancel{60}_{1}}\ ms^{-1}$ and $r = 90$

$$\tan\alpha = \frac{15 \times 15}{90 \times 10}$$

$$= \frac{225}{900}$$

$$= \frac{1}{4}$$

$$\alpha = \tan^{-1}\left(\frac{1}{4}\right).$$

$\sqrt{17}$, $\alpha$, 4, 1

(iii). Now $$F = m\cos\alpha\left(\frac{v^2}{r} - g\tan\alpha\right) \quad *$$

$\cos\alpha = \frac{4}{\sqrt{17}}$

$$F = 1200 \cdot \frac{4}{\sqrt{17}}\left(\left(\frac{\cancel{72}^{2} \times 10\cancel{00}}{\cancel{60}_{1} \times \cancel{60}_{1}}\right)^2 \cdot \frac{1}{90} - \frac{10}{4}\right)$$

$$= \frac{4800}{\sqrt{17}}\left(\frac{400}{90} - \frac{10}{4}\right)$$

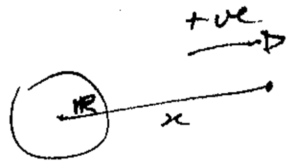
$$= \frac{4800}{\sqrt{17}}\left(\frac{800 - 450}{180}\right)$$

$$= \frac{4800}{\sqrt{17}} \times \frac{350}{180}$$

$$\doteq 2263.665 \ldots$$

$$= 2263.7 \text{ N}$$

Q6 (b) (i) $\ddot{x} \propto \frac{1}{x^2}$




So $\ddot{x} = -\frac{k}{x^2}$

Now at the earth's surface

$g = \frac{k}{R^2}$

Assume that when $t = 0$, $v = u$, $x = R$.

$k = gR^2$

So $\ddot{x} = -\frac{gR^2}{x^2}$

$\frac{d(\frac{1}{2}v^2)}{dx} = -gR^2x^{-2}$

Integrate w.r.to $x$ $\quad \frac{1}{2}v^2 = +gR^2x^{-1} + c$

$v^2 = +\frac{2gR^2}{x} + c$

For $x = R$, $v = u$, $\quad u^2 = +\frac{2gR^2}{R} + c$

$c = u^2 - 2gR$

and $\quad v^2 = u^2 - 2gR + \frac{2gR^2}{x}$

$v^2 = u^2 - 2gR^2\left(\frac{1}{R} - \frac{1}{x}\right)$

(ii) Now at greatest height $H$, $v = 0$

So $\quad 2gR^2\left(\frac{1}{R} - \frac{1}{x}\right) = u^2$

$\frac{1}{R} - \frac{1}{x} = \frac{u^2}{2gR^2}$

$\frac{1}{x} = \frac{1}{R} - \frac{u^2}{2gR^2}$

$\frac{1}{x} = \frac{2gR - u^2}{2gR^2}$

$x = \frac{2gR^2}{2gR - u^2}$

So $\quad H = \frac{2gR^2}{2gR - u^2} - R$

$H = \frac{\cancel{2gR^2} - \cancel{2gR^2} + u^2R}{2gR - u^2}$

$H = \frac{u^2R}{2gR - u^2}$

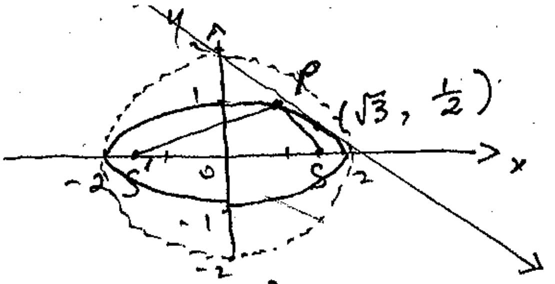
(iii) As $x \to \infty$, $v^2 \to u^2 - 2gR$ NOTE: $\frac{1}{x} \to 0^+$

so $u^2 - 2gR > 0$

$u^2 > 2gR$

$u > \sqrt{2gR}$

Q7. (a) (i)




(ii) Since $\frac{x^2}{4} + y^2 = 1$, $a = 2$ and $b = 1$

$a^2e^2 = a^2 - b^2$, So $4e^2 = 4 - 1$

$e = \frac{\sqrt{3}}{2}$

Focii are: $S'(-\sqrt{3}, 0)$, $S(\sqrt{3}, 0)$.

(iii) At $x = \sqrt{3}$, $y = \frac{1}{2}$ and equation Tangent is $y - \frac{1}{2} = m(x - \sqrt{3})$

Since $\frac{x^2}{4} + y^2 = 1$

differentiating $\frac{2x}{4} + 2y \cdot \frac{dy}{dx} = 0$

$\frac{dy}{dx} = -\frac{2x}{4} \cdot \frac{1}{2y} = -\frac{x}{4y}$

At $x = \sqrt{3}$, $m = \frac{-\sqrt{3}}{2}$

Equat. tangent is $y - \frac{1}{2} = -\frac{\sqrt{3}}{2}(x - \sqrt{3})$

$2y - 1 = -\sqrt{3}x + 3$

So $\sqrt{3}x + 2y - 4 = 0$.

(iv) Perimeter $\Delta S'PS = PS' + PS + S'S$

$= 2a + 2ae$

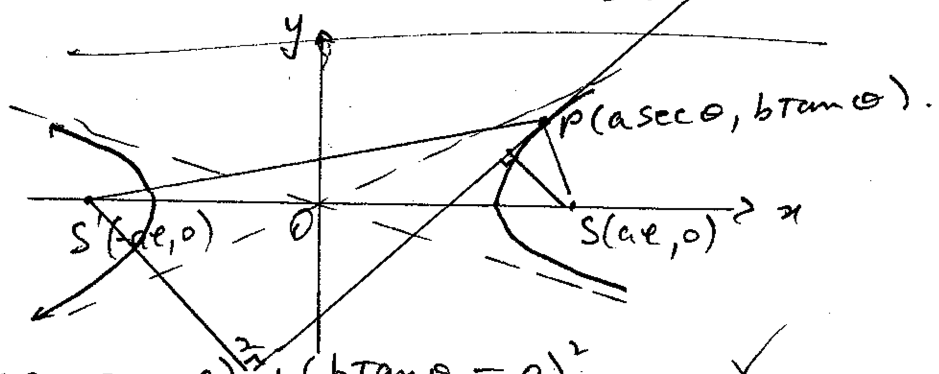
$= 2a(1 + e)$

$= 4(1 + \frac{\sqrt{3}}{2})$

$= 2(2 + \sqrt{3})$.

(b)

(i)




$SP^2 = (a\sec\theta - ae)^2 + (b\tan\theta - 0)^2$

$= a^2\sec^2\theta - 2a^2e\sec\theta + a^2e^2 + b^2\tan^2\theta$

$= a^2\sec^2\theta - 2a^2e\sec\theta + a^2e^2 + b^2\sec^2\theta - b^2$

$= (a^2 + b^2)\sec^2\theta - 2a^2e\sec\theta + a^2$ NOTE: $a^2e^2 = a^2 + b^2$

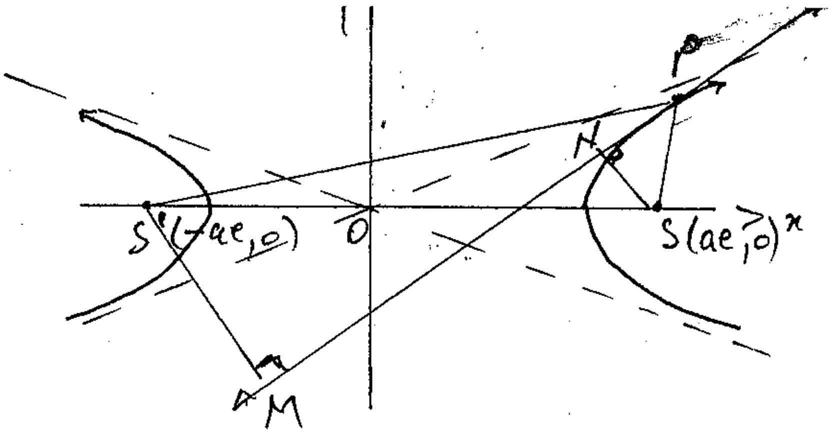
$= a^2e^2\sec^2\theta - 2a^2e\sec\theta + a^2$

$= a^2(e\sec\theta - 1)^2$

$SP = a(e\sec\theta - 1)$ and similarly $S'P = a(e\sec\theta + 1)$

OVER

Q 7(b) ii




Since equation tangent is $\dfrac{(\sec\theta)x}{a} - \dfrac{(\tan\theta)y}{b} - 1 = 0$

Perp. distance $S'M = \left| \dfrac{-\dfrac{ae\sec\theta}{a} - 0 - 1}{\sqrt{\dfrac{\sec^2\theta}{a^2} + \dfrac{\tan^2\theta}{b^2}}} \right|$ ✓

$= \left| \dfrac{-e\sec\theta - 1}{\frac{1}{ab}\sqrt{\sec^2\theta + \tan^2\theta}} \right|$ ✓

$= + \dfrac{(e\sec\theta + 1)ab}{\sqrt{\sec^2\theta + \tan^2\theta}}$

Similarly for distance $SM = \left| \dfrac{\dfrac{ae\sec\theta}{a} - 0 - 1}{\sqrt{\dfrac{\sec^2\theta}{a^2} + \dfrac{\tan^2\theta}{b^2}}} \right|$ ✓

$= \dfrac{(e\sec\theta - 1)ab}{\sqrt{\sec^2\theta + \tan^2\theta}}$

So $\dfrac{S'M}{SM} = \dfrac{e\sec\theta + 1}{e\sec\theta - 1} = \dfrac{S'P}{SP}$ ✓

and $\dfrac{S'M}{S'P} = \dfrac{SM}{SP}$ ✓

and so $\sin\angle S'PM = \sin\angle SPN$

$\angle S'PM = \angle SPN$ ✓

Q8. (a) (i) RHS $= \int_0^a f(a-x)\,dx$ Let $u = a-x$, $\frac{du}{dx} = -1$, $-du = dx$. For $x=0$, $u=a$; For $x=a$, $u=0$

$= -\int_a^0 f(u)\,du$

$= \int_0^a f(u)\,du$

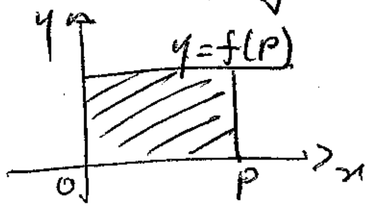
$= \int_0^a f(x)\,dx$.

(ii) Given $f(x) + f(P-x) = f(P)$ – a constant.

$$\int_0^P f(x)\,dx + \int_0^P f(P-x)\,dx = \int_0^P f(P)\,dx$$

and by part (i) $2\int_0^P f(x)\,dx = \int_0^P f(P)\,dx$

The definite integral $\int_0^P f(P)\,dx$ can be evaluated by determining the area under the curve $y = f(P)$ between $x = 0$ and $x = P$.




Area is $P\,f(P)$

and so $2\int_0^P f(x)\,dx = P\,f(P)$

$\int_0^P f(x)\,dx = \frac{1}{2} P\,f(P)$.

(iii) $I = \int_0^\pi \frac{x\,dx}{4+\sin^2 x} = \int_0^\pi \frac{(\pi - x)\,dx}{4+\sin^2(\pi - x)}$

$= \int_0^\pi \frac{\pi\,dx}{4+\sin^2 x} - \int_0^\pi \frac{x\,dx}{4+\sin^2 x}$

and so $2I = \pi \int_0^\pi \frac{dx}{4+\sin^2 x}$

$= \pi \int_0^\pi \frac{\sec^2 x\,dx}{4\sec^2 x + \tan^2 x}$ (DIVIDING BY $\cos^2 x$ TOP + BOTTOM)

$= \pi \int_0^\pi \frac{\sec^2 x\,dx}{4 + 5\tan^2 x}$ Let $u = \tan x$, $\frac{du}{dx} = \sec^2 x$, $du = \sec^2 x\,dx$

$= \frac{2\pi}{5} \int_0^{\frac{\pi}{2}} \frac{\sec^2 x\,dx}{\frac{4}{5} + \tan^2 x}$ NOTE: EVEN FUNCTION. For $x=0$, $u=0$; For $x=\frac{\pi}{2}$, $u=\infty$

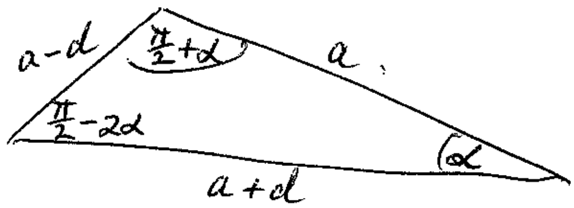
$= \frac{2\pi}{5} \int_0^\infty \frac{du}{\left(\frac{2}{\sqrt{5}}\right)^2 + u^2}$

$= \frac{2\pi}{5} \cdot \frac{\sqrt{5}}{2} \left[\tan^{-1} \frac{\sqrt{5}u}{2}\right]_0^\infty$

$= \frac{\pi}{\sqrt{5}} \left[\tan^{-1}(\infty) - \tan^{-1}(0)\right]$

$2I = \frac{\pi}{\sqrt{5}} \times \frac{\pi}{2}$ and $I = \frac{\pi^2}{4\sqrt{5}}$

Q8. (b)




BY THE SIN RULE :

$$\frac{a}{\sin(\frac{\pi}{2}-2\alpha)} = \frac{a+d}{\sin(\frac{\pi}{2}+\alpha)}$$

$$\frac{a}{\cos 2\alpha} = \frac{a+d}{\cos\alpha}.$$

$$\frac{a}{a+d} = \frac{\cos 2\alpha}{\cos\alpha} = \frac{2\cos^2\alpha - 1}{\cos\alpha}$$

$$\frac{a}{a+d} = 2\cos\alpha - \frac{1}{\cos\alpha}. \quad \text{—— (1).}$$

BY THE COSINE RULE :

$$\cos\alpha = \frac{a^2+(a+d)^2-(a-d)^2}{2a(a+d)}$$

$$= \frac{a^2+[((a+d)-(a-d))((a+d)+(a-d))]}{2a(a+d)}$$

$$= \frac{a^2+[(2d)(2a)]}{2a(a+d)}$$

$$\cos\alpha = \frac{a+4d}{2(a+d)}$$

SUBSTITUTE IN (1)

$$\frac{a}{a+d} = \frac{a+4d}{a+d} - \frac{2(a+d)}{a+4d}$$

$$a(a+4d) = (a+4d)^2 - 2(a+d)^2$$

$$\cancel{a^2}+4ad = \cancel{a^2}+8ad+16d^2-2a^2-4ad-2d^2$$

$$2a^2 = 14d^2$$

$$d^2 = \frac{1}{7}a^2$$

$$d = \frac{1}{\sqrt{7}}a$$

SIDES IN RATIO $\quad a-\frac{1}{\sqrt{7}}a : a : a+\frac{1}{\sqrt{7}}a$

$$1-\frac{1}{\sqrt{7}} : 1 : 1+\frac{1}{\sqrt{7}}$$

$$\sqrt{7}-1 : \sqrt{7} : \sqrt{7}+1$$