Name: ____________________

St George Girls High School

Trial Higher School Certificate Examination

2014

Mathematics Extension 2

General Instructions

- Reading time – 5 minutes
- Working time – 3 hours
- Write using blue or black pen.
- Write your student number on each booklet.
- Board-approved calculators may be used.
- A table of standard integrals is provided at the back of this paper.
- The mark allocated for each question is listed at the side of the question.
- Marks may be deducted for careless or poorly presented work.

Total Marks – 100

Section I – Pages 2 – 6
10 marks

- Attempt Questions 1 – 10.
- Allow about 15 minutes for this section.
- Answer on the sheet provided.

Section II – Pages 7 – 14
90 marks

- Attempt Questions 11 – 16.
- Allow about 2 hours 45 minutes for this section.
- Begin each question in a new booklet.
- Show all necessary working in Questions 11 – 16.
- Templates for Q13(a) to be detached and placed in Q13 answer booklet.

Students are advised that this is a Trial Examination only and does not necessarily reflect the content or format of the Higher School Certificate Examination.

Section I

10 marks
Attempt Questions 1 – 10
Allow about 15 minutes for this section **Marks**

Use the multiple-choice answer sheet for Questions 1–10.

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1. Which expression is equal to $\int \cos^3 x \, dx$?

   (A) $\frac{1}{4}\sin^4 x + C$

   (B) $\sin x - \frac{\sin^3 x}{3} + C$

   (C) $\sin x + \frac{\cos^3 x}{3} + C$

   (D) $\cos x - \frac{\sin^3 x}{3} + C$

2. The eccentricity of the hyperbola with the equation $\frac{x^2}{3} - \frac{y^2}{4} = 1$ is:

   (A) $1 + \frac{2}{\sqrt{3}}$

   (B) $\sqrt{\frac{7}{3}}$

   (C) $\frac{\sqrt{21}}{3}$

   (D) $\frac{5}{3}$

3. Let the point R represent the complex number z on an Argand diagram. Which of the following describes the locus of R specified by $2|z| = z + \bar{z} + 4$.

   (A) Circle with centre $(0,0)$ and radius 4

   (B) Parabola with vertex $(-1,0)$, axis $y = 0$

   (C) Parabola with vertex $(-1,0)$, axis $x = 0$

   (D) Perpendicular bisector of $(0,0)$ and $(0,4)$

Section I (cont'd)

4. The polynomial $P(x) = x^4 + ax^3 - bx^2 - 12x$ has a double root at $x = -2$. What are the values of a and b.

   (A) $a = -1$ and $b = -8$

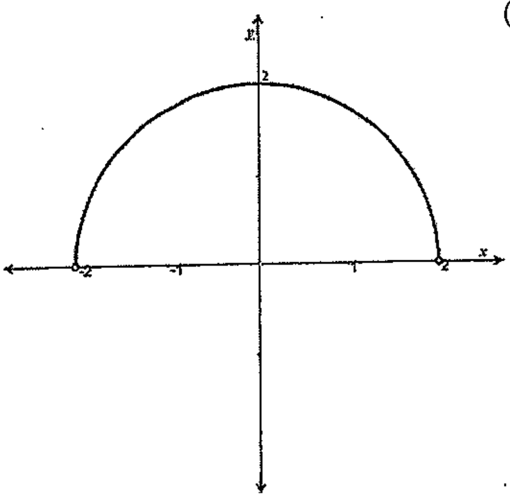
   (B) $a = 8$ and $b = 1$

   (C) $a = 2$ and $b = 4$

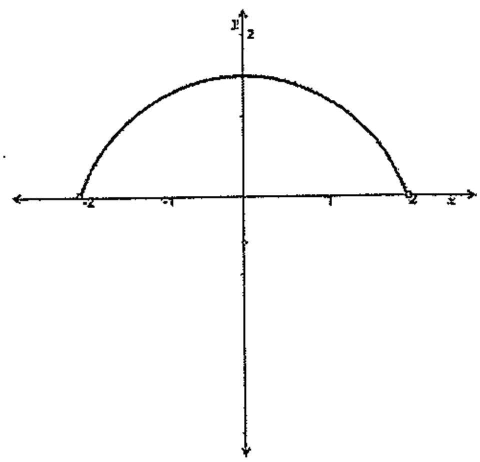
   (D) $a = 1$ and $b = 8$

5. The locus of z if $\arg(z-2) - \arg(z+2) = \dfrac{\pi}{4}$ is best shown as:

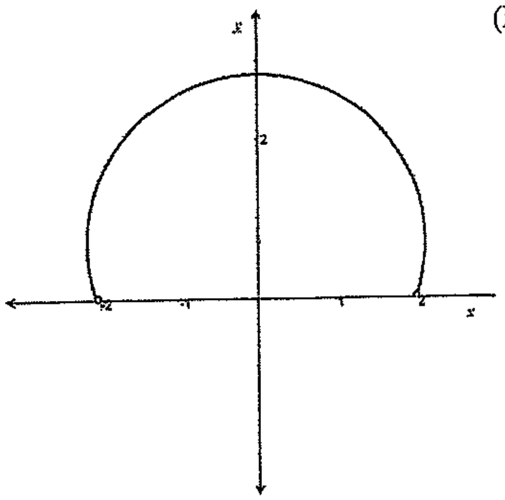
(A)



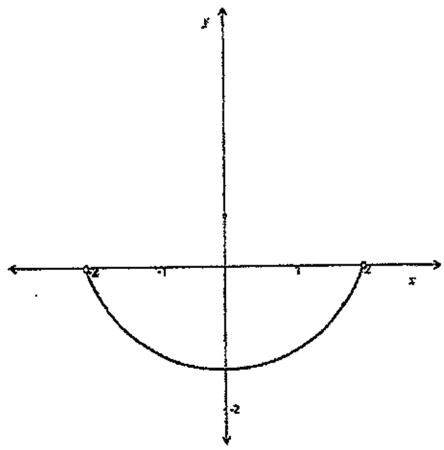
(B)



(C)



(D)



Section I (cont'd)

6. The derivative of the curve $x^3 + 9x^2 - y^2 + 27x - 4y + 23 = 0$ is:

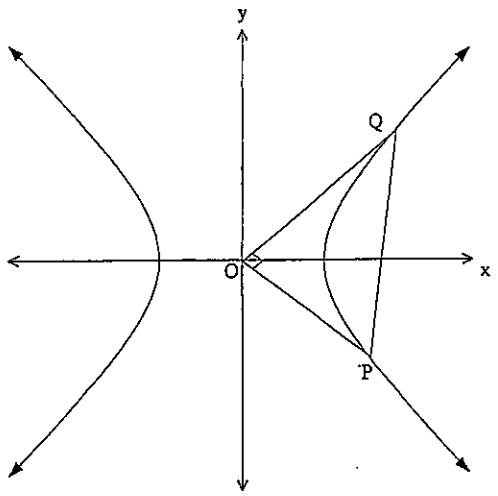
(A) $\frac{dy}{dx} = \frac{x^2+6x+9}{2y}$

(B) $\frac{dy}{dx} = \frac{3x^2+18x+27}{2y+4}$

(C) $\frac{dy}{dx} = \frac{3x^2+18x+27}{-(2y+)4}$

(D) $\frac{dy}{dx} = \frac{x^2+6x+9}{-2y}$

7. The diagram below shows the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ where $a > b > 0$. The points $P(a\sec\theta,\ b\tan\theta\ )$ and $Q(a\sec\alpha,\ b\tan\alpha\ )$ lie on the hyperbola and the chord PQ subtends a right angle at the origin.



Use the parametric representation of the hyperbola to determine which of the following expressions is correct?

(A) $\sin\theta\sin\alpha = -\frac{a^2}{b^2}$

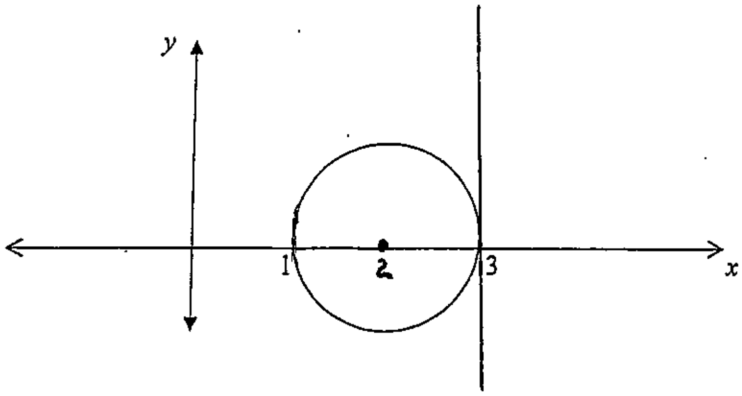
(B) $\sin\theta\sin\alpha = \frac{a^2}{b^2}$

(C) $\tan\theta\tan\alpha = -\frac{a^2}{b^2}$

(D) $\tan\theta\tan\alpha = \frac{a^2}{b^2}$

Section I (cont'd)

8.



In the diagram above the circle $(x-2)^2+y^2=1$ is shown. The region bounded by the circle is rotated about the line $x=3$. Using the method of cylindrical shells the volume of the solid of revolution so formed is given by:

(A) $V=4\pi\int_1^3 (x-3)\sqrt{1-(x-2)^2}\,dx$

(B) $V=4\pi\int_2^3 (3-x)\sqrt{1-(x-2)^2}\,dx$

(C) $V=2\pi\int_1^3 (3-x)\sqrt{1-(x-2)^2}\,dx$

(D) $V=2\pi\int_1^3 (3-x)\sqrt{1-(x-2)^2}\,dx$

9. Let $P(x)$ be a polynomial of degree $n>0$ such that $P(x)=(x-\alpha)^p\,.\,Q(x)$, where $p\geq 2$ and α is a real number. $Q(x)$ is a polynomial with real coefficients of degree $q>0$. Which of the following is <u>definitely</u> an incorrect statement?

(A) $P(x)$ changes sign around the root $x=\alpha$

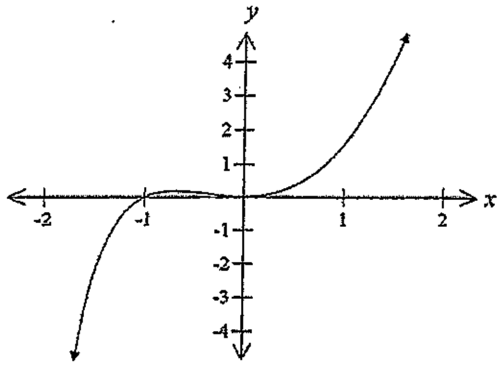
(B) $n\leq p+q$

(C) Roots of $Q(x)$ are conjugates of one another

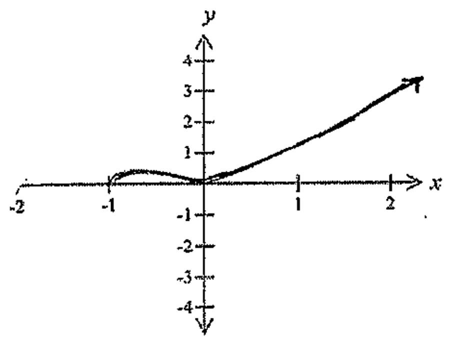
(D) $P'(\alpha)>0$

Section I (cont'd)

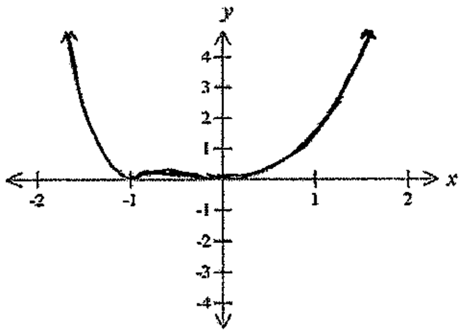
10. The diagram shows the graph of the function $y = f(x)$.



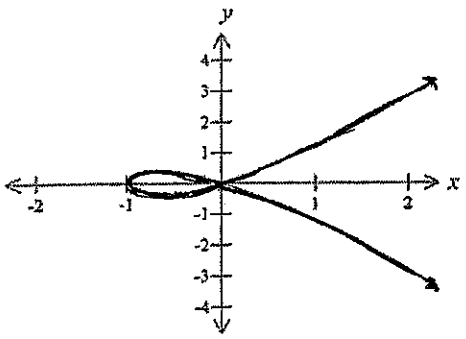
The diagram that shows the graph of the function $y = [f(x)]^2$ is:



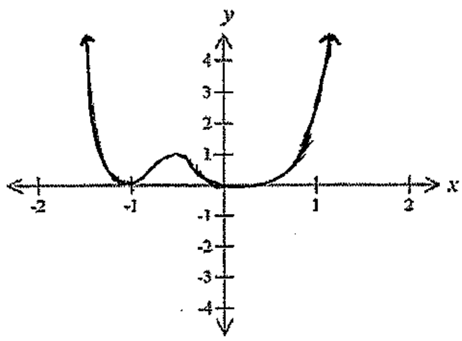
(B)



(C)



(D)



End of Section I

Section II

90 marks
Attempt Questions 11 – 16
Allow about 2 hours 45 minutes for this section

Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

In Questions 11 – 16, your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (15 marks) Use a SEPARATE writing booklet **Marks**

a) Given $z = \dfrac{1-i}{\sqrt{3}+i}$.

(i) Find the modulus $|z|$ and argument $\arg(z)$ of z. 2

(ii) Find the smallest positive integer n such that z^n is REAL. 2

b) Find the complex square roots of $1 - 2\sqrt{2}i$ giving your answers in the form $x + iy$ where x and y are real. 3

c) Find the three different values of z for which $z^3 = \dfrac{1}{\sqrt{2}} + \dfrac{1}{\sqrt{2}}i$. 3

d) (i) On an Argand Diagram, draw and shade the region R given by 1

$$|z - 3 - 3i| \leq 3 .$$

(ii) P is a point in the region R, representing the complex number z. 2

What is the maximum value of $|z|$?

(iii) The tangent to the curve at P cuts the x-axis at the point T. By using the nature of ΔOPT, or otherwise, find the exact area of ΔOPT. 2

Question 12 (15 marks) Use a SEPARATE writing booklet **Marks**

a) Using the substitution $t = \tan\frac{x}{2}$, evaluate $\displaystyle\int_0^{\frac{\pi}{2}} \frac{dx}{5 + 3\cos x}$. **4**

[leave your answer in EXACT form]

b) Find $\displaystyle\int \sin^4 x \cos^3 x \; dx$. **3**

c) (i) Show that $(1 + t^2)^{n-1} + t^2\,(1 + t^2)^{n-1} = (1 + t^2)^n$. **1**

(ii) Let $I_n = \displaystyle\int_0^x (1 + t^2)^n \; dt$ for n a positive integer. **4**

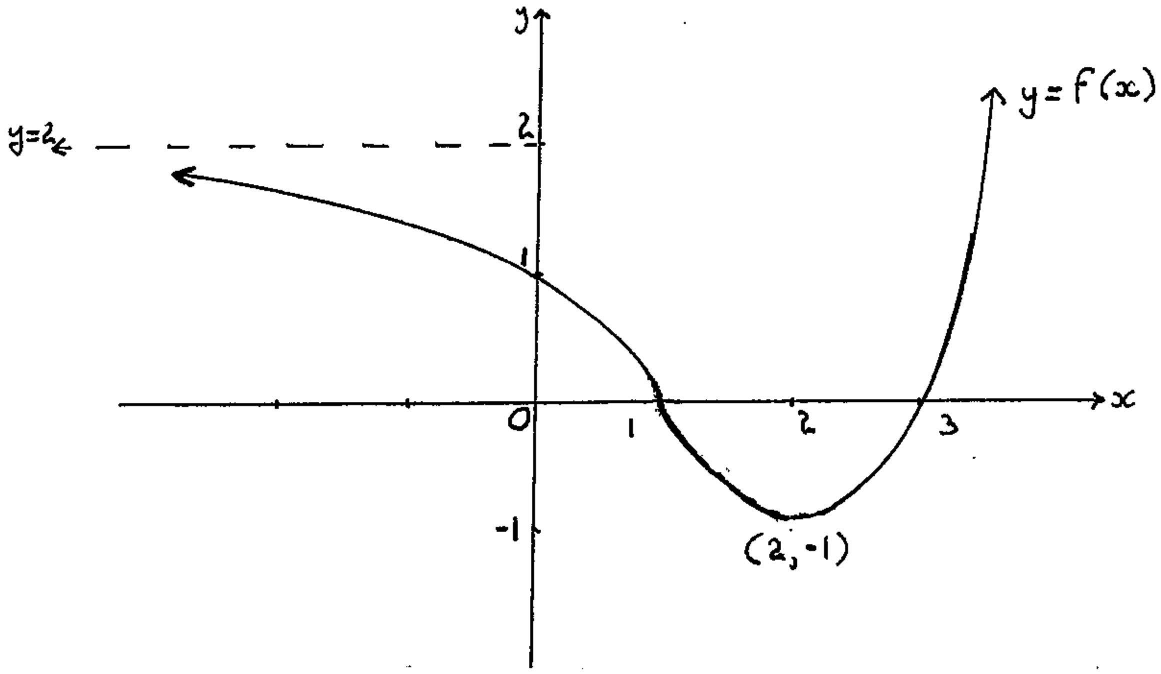
Use integration by parts, and part (i) above, to show that

$$I_n = \frac{1}{2n+1}\, x(1 + x^2)^n + \frac{2n}{2n+1}\, I_{n-1}.$$

d) Make a suitable substitution to find the exact value of $\displaystyle\int_0^{\frac{1}{2}} \frac{dx}{\sqrt{x}\,.\sqrt{1-x}}$. **3**

Question 13 (15 marks) Use a SEPARATE writing booklet **Marks**

a) The diagram below shows the graph of a function $f(x)$.



Using the separate templates of the graph of $y = f(x)$ provided at the end of this paper, sketch the graphs of:

(i) $y = [f(x)]^2$. 2

(ii) $y = f'(x)$. 2

(iii) $y^2 = f(x)$. 2

Question 13 continued on next page

Question 13 (cont'd) **Marks**

b) (i) Show that the equation of the normal to the hyperbola $xy = c^2$ at $P\left(cp, \frac{c}{p}\right)$ is $p^3x - py = c(p^4 - 1)$. 2

(ii) The normal at $P\left(cp, \frac{c}{p}\right)$ meets the hyperbola $xy = c^2$ again at $Q\left(cq, \frac{c}{q}\right)$. 2

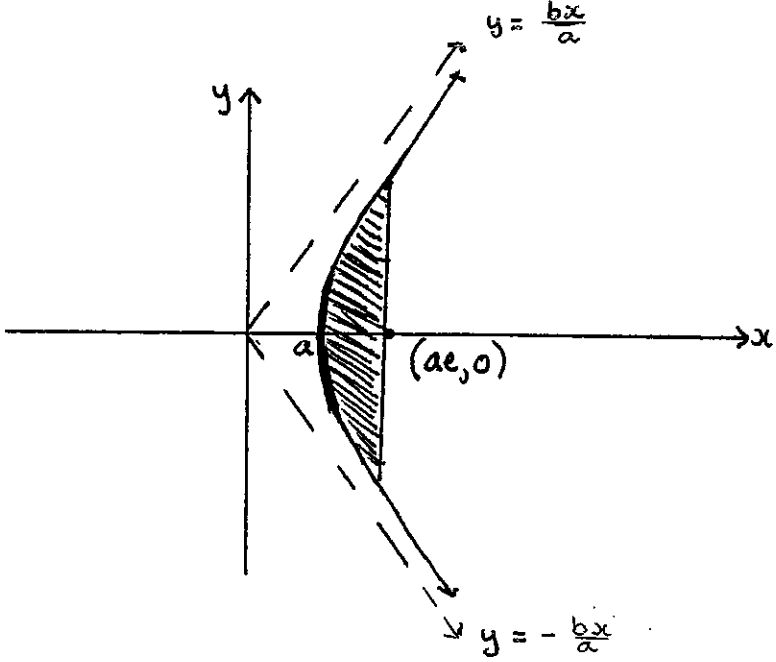
Prove that $p^3q = -1$.

(iii) Hence, show that if $M(x, y)$ is the midpoint of PQ, then $\frac{x}{y} = -\frac{1}{p^2}$. 3

c) Sketch on an Argand Diagram (at least $\frac{1}{3}$ of a page) the locus of the complex number z where $\arg(z + 1) = \arg(z - 1 + i)$. 2

Question 14 (15 marks) Use a SEPARATE writing booklet **Marks**

a) The shaded region shown below represents the area bounded by the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ and the line $x = ae$. 4



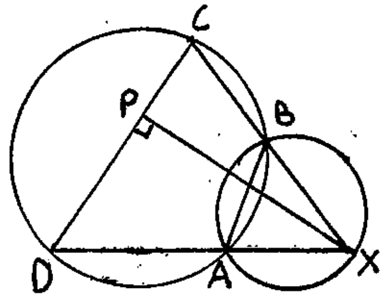
Find the volume generated by rotating this area about the y-axis through 360° (answer in terms of a, b).

b) When a polynomial $P(x)$ is divided by $(x-3)$ the remainder is -2. When $P(x)$ is divided by $(x+2)$ the remainder is 8. Find the remainder when $P(x)$ is divided by $(x-3)(x+2)$. 3

Question 14 continued on next page

Question 14 (cont'd) **Marks**

c)



In the diagram above, $AB = AD = AX$ and $XP \perp DC$.

(i) Prove that $\angle DBX = 90°$. 2

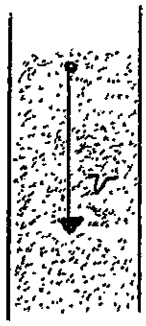
(ii) Hence, or otherwise, prove that $\angle APB = \angle ABP$. 2

d) The three non-zero roots of the equation $x^3 - 3px + q = 0$ are $\alpha,\ \beta,\ \gamma$. 4

Find the <u>monic</u> equation whose roots are $\frac{\beta\gamma}{\alpha}, \frac{\alpha\gamma}{\beta}, \frac{\alpha\beta}{\gamma}$ expressing its coefficients in terms of p and q.

Question 15 (15 marks) Use a SEPARATE writing booklet **Marks**

a) Consider a particle falling through a fluid as shown in the diagram below:



The resistive frictional force on the particle is proportional to its velocity. That is, the resistance force may be written as $R = -mkv$ where k is a constant and the particles velocity is $v(ms^{-1})$.

(i) If the particle falls vertically from rest, show that the terminal velocity v_T is given by $v_T = \frac{g}{k}$, where $g(ms^{-2})$ is the acceleration due to gravity. 3

(ii) If the particle is projected <u>upwards</u> into the resistive fluid with speed v_T, show that after t seconds.

(α) its speed $v(ms^{-1})$ is given by $v = v_T(2e^{-kt} - 1)$. 3

(β) its height, $x(m)$ is given by $x = \frac{v_T}{k}(2 - kt - 2e^{-kt})$. 3

(iii) Hence, show that the greatest height that the particle can reach is 2

$$x_{\max} = \frac{v_T}{k}(1 - \ln 2).$$

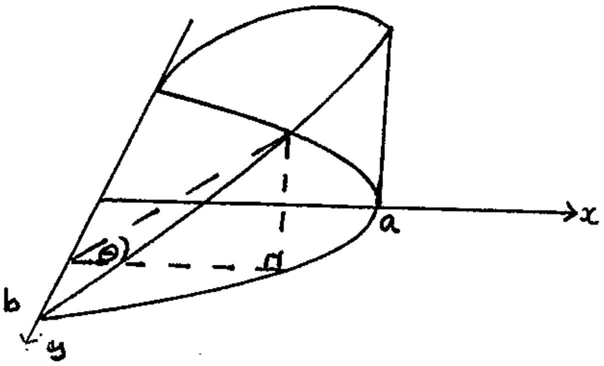
b) The equation $(\sin^2\theta)\,z^2 - (\sin 2\theta)\,z + 1 = 0$, where $0 < \theta < \frac{\pi}{2}$, has roots α and β.

(i) Show that the roots of the equation are $(\cot\theta + i)$ and $(\cot\theta - i)$. 2

(ii) Hence, show that $\alpha^n + \beta^n = \frac{2\cos n\theta}{\sin^n\theta}$. 2

Question 16 (15 marks) Use a SEPARATE writing booklet **Marks**

a) A solid in the shape of a wedge has its base in half an ellipse, with major axis $2a$ and minor axis $2b$. Cross-sections taken perpendicular to the base are all right angled triangles and the angle between the two flat surfaces of the wedge is $\theta°$.



(i) Show that the area of the triangular face of the cross-section is given by 3

$$\frac{a^2}{2b^2}(b^2 - y^2)\tan\theta .$$

(ii) Hence, or otherwise, find the volume of the wedge giving your answer in terms of a, b and $\tan\theta$. 3

b) (i) Prove that

(α) $\dfrac{{}^1C_0}{x} - \dfrac{{}^1C_1}{x+1} = \dfrac{1!}{x(x+1)}$. 2

(β) $\dfrac{{}^2C_0}{x} - \dfrac{{}^2C_1}{x+1} + \dfrac{{}^2C_2}{x+2} = \dfrac{2!}{x(x+1)(x+2)}$.

(ii) Given $T(k,x) = \dfrac{k!}{x(x+1)(x+2)...(x+k)}$, prove that 3

$T(k,x) - T(k,x+1) = T(k+1,x)$.

(iii) Hence prove, using Mathematical Induction or otherwise, that for $n \geq 1$: 4

$$\frac{{}^nC_0}{x} - \frac{{}^nC_1}{x+1} + \frac{{}^nC_2}{x+2} - \frac{{}^nC_3}{x+3} + ... + (-1)^n\frac{{}^nC_n}{x+n} = \frac{n!}{x(x+1)(x+2)(x+3)...(x+n)}.$$

[You may use the result: ${}^{k+1}C_r = {}^kC_r + {}^kC_{r-1}$]

[Note: ${}^{k+1}C_0 = {}^kC_0$ and ${}^{k+1}C_{k+1} = {}^kC_k$]

End of Paper

Student Number: ______________________ Class Teacher: ____________________

Section I

Year 12 Trial HSC Examination 2014

Mathematics Extension 2

Multiple-choice Answer Sheet - Questions 1 – 10

Select the alternative A, B, C or D that best answers the question. Fill in the response oval completely.

Sample $2 + 4 =$ (A) 2 (B) 6 (C) 8 (D) 9

A ○ B ● C ○ D ○

If you think you have made a mistake, put a cross through the incorrect answer and fill in the new answer.

A ● B ⊗ C ○ D ○

If you change your mind and have crossed out what you consider to be the correct answer, then indicate this by writing the word *correct* and drawing an arrow as follows:

correct

A ⊗ B ⊗ C ○ D ○

---

1. A ○ B ○ C ○ D ○
2. A ○ B ○ C ○ D ○
3. A ○ B ○ C ○ D ○
4. A ○ B ○ C ○ D ○
5. A ○ B ○ C ○ D ○
6. A ○ B ○ C ○ D ○
7. A ○ B ○ C ○ D ○
8. A ○ B ○ C ○ D ○
9. A ○ B ○ C ○ D ○
10. A ○ B ○ C ○ D ○

Staff Use Only

Section I		/10
Section II		/90
Question 11	/15	
Question 12	/15	
Question 13	/15	
Question 14	/15	
Question 15	/15	
Question 16	/15	
Total		/100

EXT 2 SOLUTIONS TRIAL 2014

Section I

Q1 $\int \cos x \left(1 - \sin^2 x\right) dx$

$= \int \left[\cos x - \cos x . \sin^2 x\right] dx$ (B)

$= \sin x - \frac{1}{3}\sin^3 x + C$

Q2 $e^2 = 1 + \frac{b^2}{a^2}$

$e = \sqrt{1 + \frac{4}{3}}$ (B)

or (C)

$= \sqrt{\frac{7}{3}} = \frac{\sqrt{21}}{3}$

Q3 $2\sqrt{x^2+y^2} = 2x+4$

$x^2 + y^2 = x^2 + 4x + 4$ (B)

$y^2 = 4(x+1)$

Q4 $P'(x) = 4x^3 + 3ax^2 - 2bx - 12$

$P'(2) = {}^-32 + 12a + 4b - 12$

$\therefore \quad 12a + 4b = 44$ (D)

$3a + b = 11 \quad \text{---(I)}$

$P(-2) = 16 - 8a - 4b + 24$

$\therefore \quad 8a + 4b = 40$

$2a + b = 10 \quad \text{---(II)}$

(I) - (II) $a = 1 \quad \therefore b = 8$

Q5. (C)

Q6. $3x^2 + 18x - 2y.\frac{dy}{dx} + 27 - 4.\frac{dy}{dx} = 0$

$\therefore \quad (2y+4)\frac{dy}{dx} = 3x^2 + 18x + 27$ (B)

$\frac{dy}{dx} = \frac{3x^2 + 18x + 27}{2y+4}$

Q7 $m_1 = \frac{b \tan\alpha}{a \sec\alpha}$ & $m_2 = \frac{b\tan\theta}{a\sec\theta}$

$\perp \Rightarrow m_1 \times m_2 = -1$

$\therefore \quad \frac{b^2 \tan\alpha \tan\theta}{a^2 \sec\theta \sec\alpha} = -1$

$\frac{b^2}{a^2} = -\frac{\sec\alpha\sec\theta}{\tan\alpha\tan\theta}$

$-\frac{b^2}{a^2} = \frac{\frac{1}{\cos\alpha\cos\theta}}{\frac{\sin\alpha\sin\theta}{\cos\alpha\cos\theta}}$ (A)

$-\frac{b^2}{a^2} = \frac{1}{\sin\alpha \ldots \theta}$

Q8 $\delta V = (3-x).2\pi.2\sqrt{1-(x-2)^2}\,dx$ (D)

$\therefore \quad V = 4\pi\int_1^3 (3-x)\sqrt{1-(x-2)^2}\,dx$

Q9 (D)

Q10. (B)

Section 2
Question Number: 11

(a)(i) $|z| = \dfrac{|1-i|}{|\sqrt{3}+i|}$ or $z = \dfrac{1-i}{\sqrt{3}+i} \times \dfrac{\sqrt{3}-i}{\sqrt{3}-i}$

$= \dfrac{\sqrt{1^2+(-1)^2}}{\sqrt{(\sqrt{3})^2+1^2}}$ $\qquad = \dfrac{(\sqrt{3}-1) - i(\sqrt{3}+1)}{4}$

$= \dfrac{\sqrt{2}}{2}$ $\qquad |z| = \sqrt{\left[\dfrac{(\sqrt{3}-1)^2}{4^2} + \dfrac{(\sqrt{3}+1)^2}{4^2}\right]}$

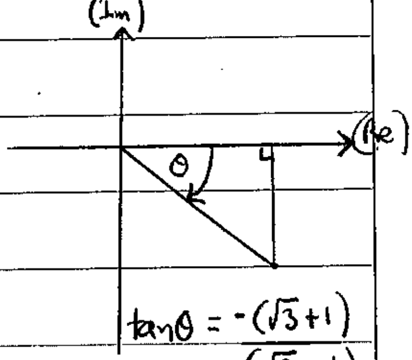
$= \dfrac{1}{4}\sqrt{4-2\sqrt{3}+4+2\sqrt{3}}$

$= \dfrac{\sqrt{2}}{2}$

and

$\arg(z) = \arg\left[\dfrac{(1-i)}{(\sqrt{3}+i)}\right]$

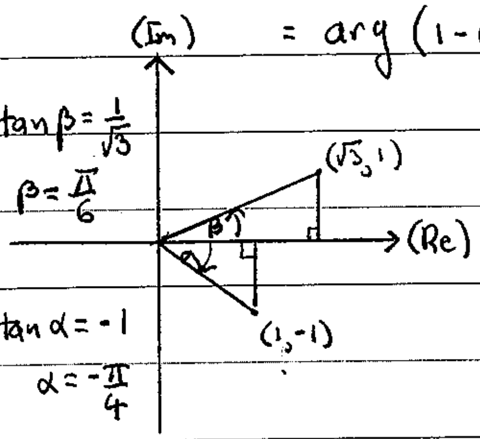
or $z = \dfrac{(\sqrt{3}-1)}{4} - \dfrac{(\sqrt{3}+1)}{4}$



$= \arg(1-i) - \arg(\sqrt{3}+i)$

$\tan\beta = \dfrac{1}{\sqrt{3}}$

$\beta = \dfrac{\pi}{6}$



$\tan\alpha = -1$

$\alpha = -\dfrac{\pi}{4}$

* $\arg(z) = \left(-\dfrac{\pi}{4}\right) - \left(\dfrac{\pi}{6}\right)$

$= \dfrac{-5\pi}{12}$

$\tan\theta = \dfrac{-(\sqrt{3}+1)}{(\sqrt{3}-1)}$

$= -(2+\sqrt{3})$

* $= -\dfrac{5\pi}{12}$

(ii) $z = \dfrac{\sqrt{2}}{2}\,\text{cis}\left(\dfrac{-5\pi}{12}\right)$

then $z^n = \left(\dfrac{\sqrt{2}}{2}\right)^n \text{cis}\left(\dfrac{-5\pi}{12}\right)^n$

De Moivre's Theorem

$= \left(\dfrac{\sqrt{2}}{2}\right)^n \left(\cos\left(\dfrac{-5\pi n}{12}\right) + i\sin\left(\dfrac{-5\pi n}{12}\right)\right)$

$= \left(\dfrac{\sqrt{2}}{2}\right)^n \left(\cos\left(\dfrac{5\pi n}{12}\right) - i\sin\left(\dfrac{5\pi n}{12}\right)\right)$

Require $\sin\left(\frac{5n\pi}{12}\right) = 0$ if z^n is REAL

$\therefore \frac{5n\pi}{12} = 0, \pi, 2\pi \ldots$

$\frac{5n}{12} = 0, 1, 2 \ldots$ Req n to be a multiple of 12

When $n = 12$, $z^{12} = (2^{-\frac{1}{2}})^{12} \cdot (\cos(5\pi) - i\sin(5\pi))$

$= 2^{-6} \cdot 1$

$= -\frac{1}{64}$

(b) Let $(x+iy)^2 = 1 - i \cdot 2\sqrt{2}$

then $x^2 - y^2 + i \cdot 2xy = 1 - i\, 2\sqrt{2}$

So equating Real and Imaginary parts

$x^2 - y^2 = 1 \ldots (\mathrm{I})$

$2xy = -2\sqrt{2} \quad (\mathrm{II}) \Longrightarrow \quad x^2 - \frac{2}{x^2} = 1$

$y = -\frac{\sqrt{2}}{x} \qquad x^4 - x^2 - 2 = 0$

gives $x^2 = \frac{1 \pm \sqrt{9}}{2}$

Since x is REAL $\quad x^2 = 2$

$x = \pm\sqrt{2}$

Square root $(\sqrt{2} - i)$ and $(-\sqrt{2} + i)$

(c) z is cube root of $\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}} = w$

$|w| = 1 \qquad \therefore w = \cos\left(\frac{\pi}{4} + 2k\pi\right) + i\sin\left(\frac{\pi}{4} + 2k\pi\right)$

$\arg(w) = \frac{\pi}{4}$

Let $z = r(\cos\theta + i\sin\theta)$. $\qquad |z| = |w|$

$= 1$

then $z^3 = (\cos 3\theta + i\sin 3\theta)$ by de Moivre

Equating Real & imaginary parts

$$\cos(3\theta) = \cos\left(\frac{\pi}{4} + 2k\pi\right) \qquad \sin 3\theta = \sin\left(\frac{\pi}{4} + 2k\pi\right)$$

$$3\theta = \frac{\pi}{4} + 2k\pi$$

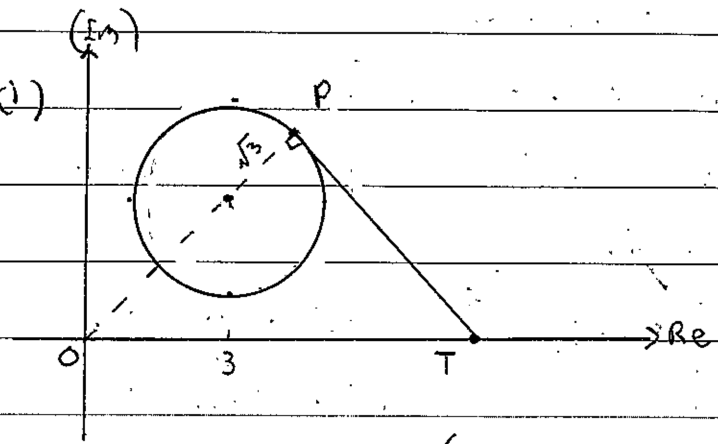
$$\therefore \theta = \frac{\pi}{12}(1 + 8k)$$

when $k=0$, $z_1 = cis\left(\frac{\pi}{12}\right) = \cos\left(\frac{\pi}{12}\right) + i\sin\left(\frac{\pi}{12}\right)$

$k=1$, $z_2 = cis\left(\frac{3\pi}{4}\right) = -\cos\left(\frac{\pi}{4}\right) + i\sin\left(\frac{\pi}{4}\right)$

$k=2$, $z_3 = cis\left(\frac{17\pi}{12}\right) = -\cos\left(\frac{5\pi}{12}\right) - i\sin\left(\frac{5\pi}{12}\right)$

(d)(i)



$|z - (3+3i)| \le 3$

Circle centre $(3,3)$

radius $\sqrt{3}$

(ii) Max. value $|z|$ is $(|3+3i| + \text{radius of circle})$

$$= \sqrt{3^2+3^2} + \sqrt{3}$$

$$= 3\sqrt{2} + \sqrt{3}$$

(iii) $\arg(z) = \arg(3+3i)$

$$= \frac{\pi}{4}$$

then ΔOPT is isosceles right angle Δ

gives $OP = OT$

$\therefore$ Area $\Delta = \frac{1}{2}(OP.OT)$

$$= \frac{1}{2}(3\sqrt{2} + \sqrt{3})^2$$

Start here for
Question Number: 12

(a) Let $t = \tan \frac{x}{2}$

then $x = 2\tan^{-1} t$

$\frac{dx}{dt} = \frac{2}{1+t^2}$

When $x = 0$, $t = 0$

$x = \frac{\pi}{2}$, $t = 1$

$\cos x = \frac{1-t^2}{1+t^2}$

On Substitution

$$\int_0^1 \frac{\frac{2}{1+t^2}\, dt}{5 + 3\frac{(1-t^2)}{1+t^2}}$$

$$= \int_0^1 \frac{2\, dt}{5(1+t^2) + 3(1-t^2)}$$

$$= \int_0^1 \frac{2\, dt}{8 + 2t^2}$$

$$= \int_0^1 \frac{dt}{4+t^2}$$

$$= \frac{1}{2}\left[\tan^{-1}\left(\frac{t}{2}\right)\right]_0^1$$

$$= \frac{1}{2}\tan^{-1}\left(\frac{1}{2}\right)$$

(b) $$\int \sin^4 x . \cos^3 x\, dx = \int \sin^4 x . (\cos^2 x) . \cos x\, dx$$

$$= \int \sin^4 x\,(1 - \sin^2 x) . \cos x\, dx$$

$$= \int \sin^4 x . \cos x\, dx - \int \sin^6 x . \cos x\, dx$$

$\left[\int f'(x)\,[f(x)]^n\, dx\right]$

$$= \frac{1}{5}\sin^5 x - \frac{1}{7}\sin^7 x + C$$

(c)(i) L.H.S. $= (1+t^2)^{n-1}\left[1 + t^2\right]$

$= (1+t^2)^n$

$=$ R.H.S.

(ii) Let $I_n = \int_0^x (1+t^2)^n \cdot \frac{d}{dt}(t) \cdot dt$

$= \left[t.(1+t^2)^n \right]_0^x - \int_0^x t.\, n.(1+t^2)^{n-1}.\, 2t \; dt$

$= x.(1+x^2)^n - 2n \int_0^x t^2 (1+t^2)^{n-1} dt$

Using part(i) $= x(1+x^2)^n - 2n \int_0^x \left[(1+t^2)^n - (1+t^2)^{n-1} \right] dt$

$I_n = x(1+x^2)^n - 2n\, I_n + 2n\, I_{n-1}$

$(1+2n) I_n = x(1+x^2)^n + 2n\, I_{n-1}$

$\therefore I_n = \dfrac{x.(1+x^2)^n}{(1+2n)} + \dfrac{2n}{(1+2n)} . I_{n-1}$

(d) Let $u^2 = 1-x$

then $x = 1-u^2$

and $2u.\, du = -dx$

Substitution gives

$\int_1^{\frac{1}{\sqrt{2}}} \dfrac{-2u.\, du}{\sqrt{1-u^2}.\, u}$

When $x=0$, $u=1$

$x = \frac{1}{2}$, $u = \frac{1}{\sqrt{2}}$

$x>0$, $u>0$

$= 2 \int_{\frac{1}{\sqrt{2}}}^1 \dfrac{du}{\sqrt{1-u^2}}$

$= 2 \left[\sin^{-1} u \right]_{\frac{1}{\sqrt{2}}}^1$

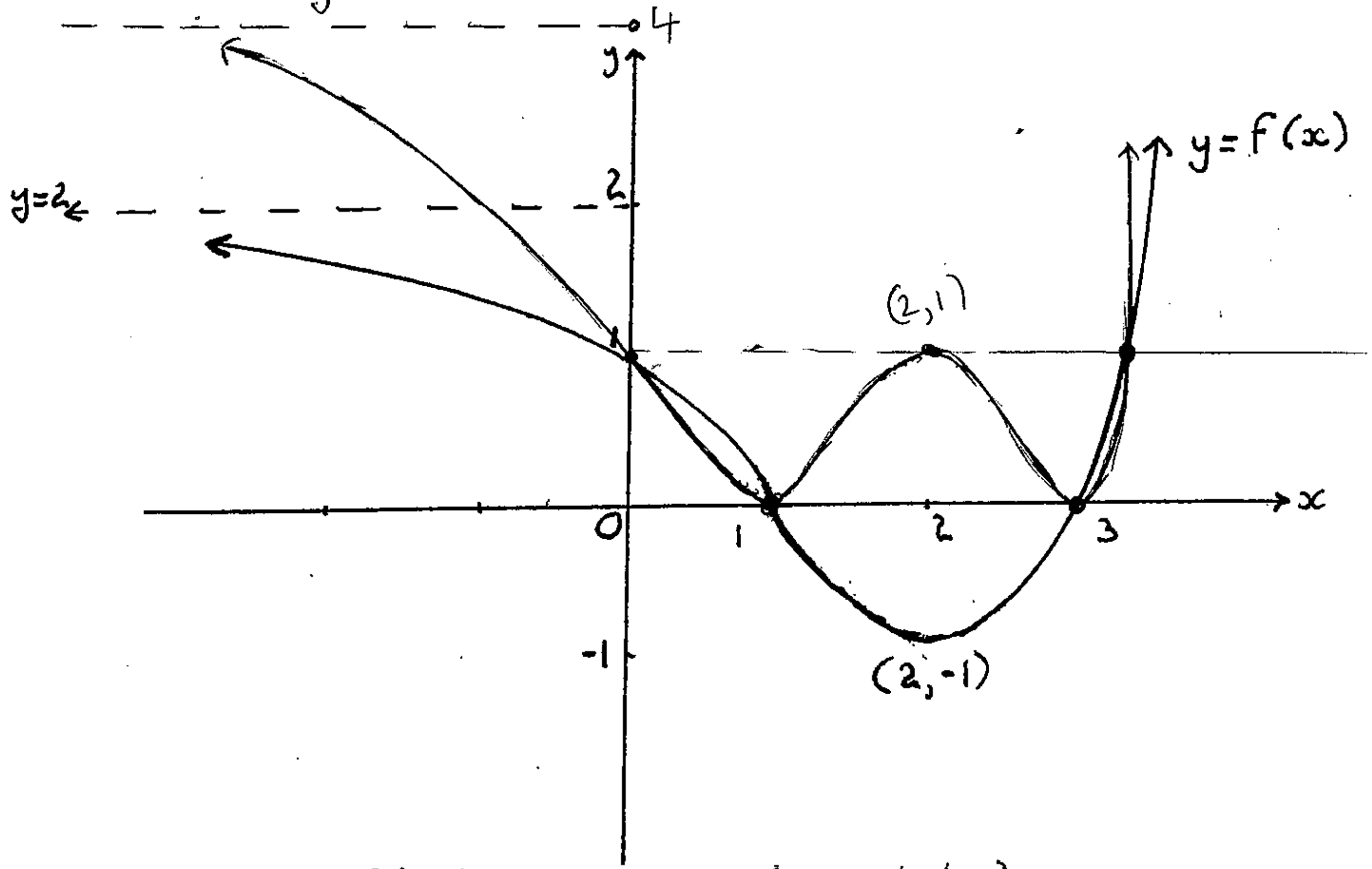
$= 2 \left[\sin^{-1} 1 - \sin^{-1} \frac{1}{\sqrt{2}} \right]$

$= 2 \left[\frac{\pi}{2} - \frac{\pi}{4} \right]$

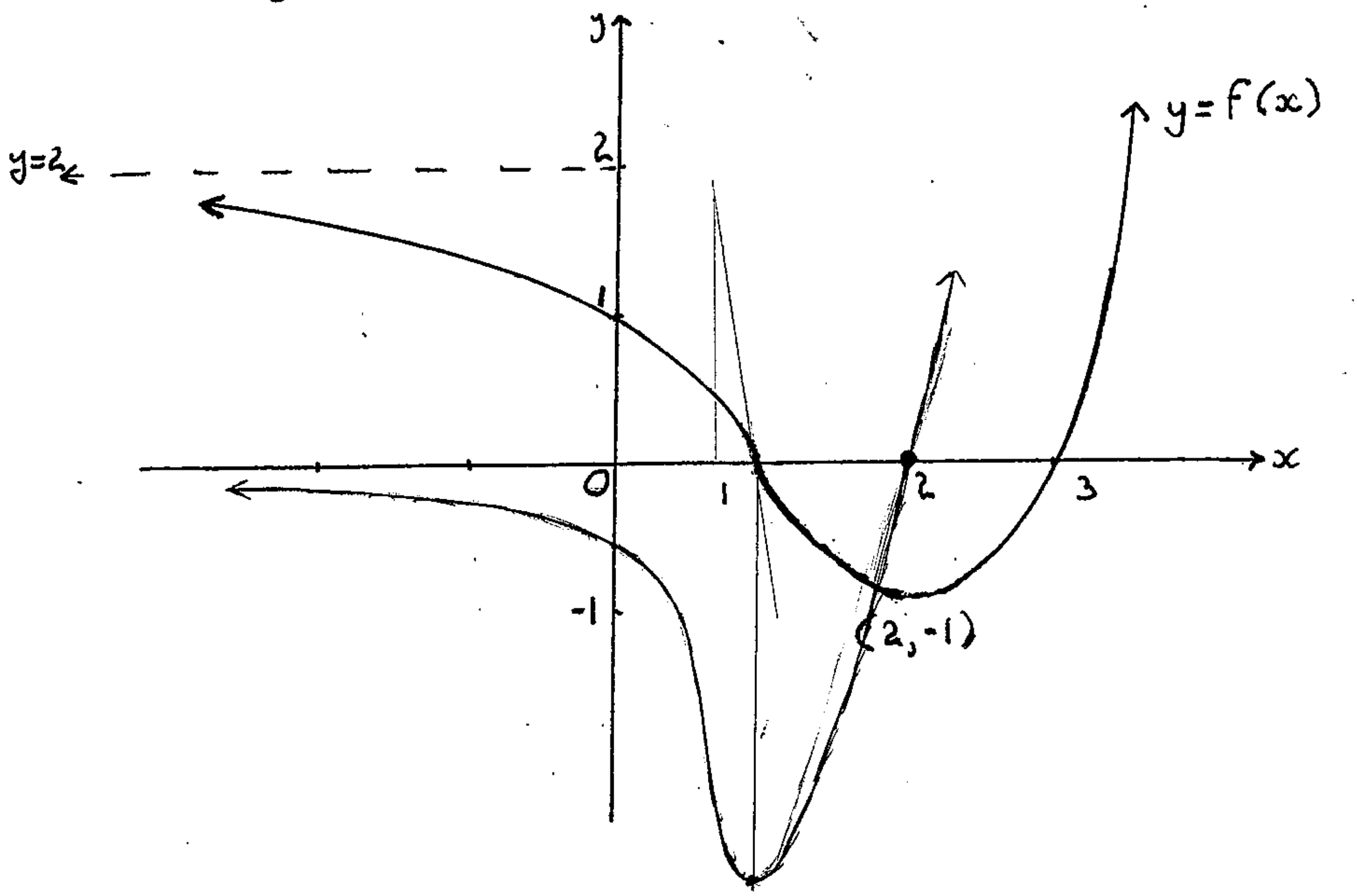
$= \frac{\pi}{2}$

QUESTION 13.

(a) (i) $y = [f(x)]^2$ (template)

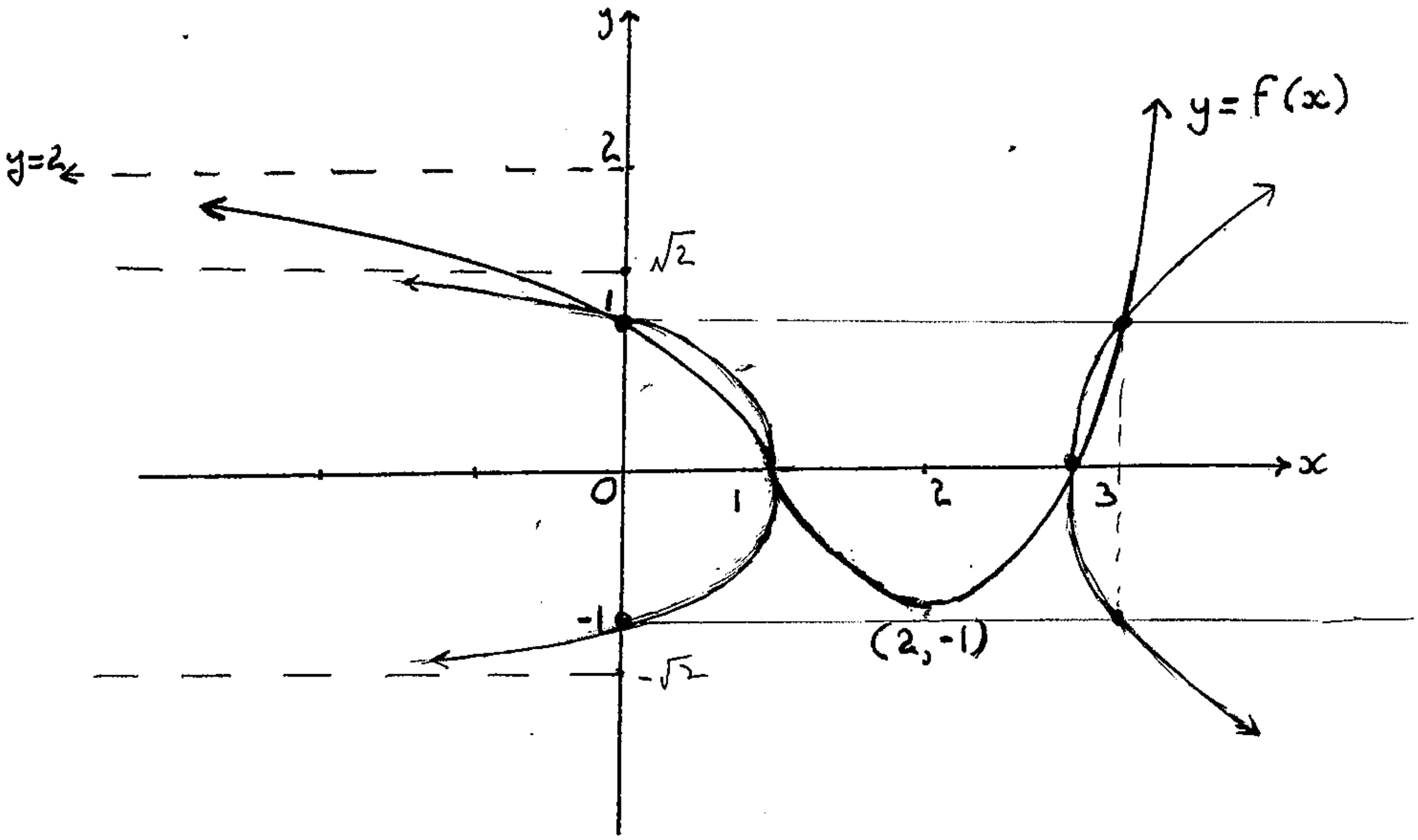


(a) (ii) $y = f'(x)$ (template)

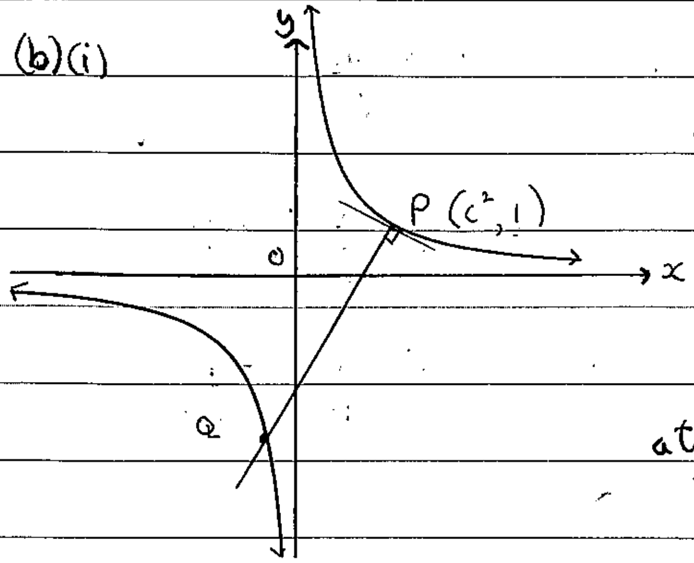


QUESTION 13.

(a) (iii) $y^2 = f(x)$ (template)



(b)(i)



$$xy = c^2 \qquad \text{or} \qquad y = \frac{c^2}{x}$$

then $y + x.\dfrac{dy}{dx} = 0$ $\qquad \dfrac{dy}{dx} = \dfrac{-c^2}{x^2}$

$$\frac{dy}{dx} = \frac{-y}{x}$$

at $P\left(cp, \frac{c}{p}\right)$, $m_{\text{tangent}} = \dfrac{-\frac{c}{p}}{cp}$

$$= -\frac{1}{p^2}$$

$\therefore$ Equation of tangent at P gradient p^2

$$y - \frac{c}{p} = p^2 (x - cp)$$

$$py - c = p^3 x - cp^4$$

$$p^3 x - py = c(p^4 - 1)$$

(ii) Q lies on hyperbola, let $Q\left(cq, \frac{c}{q}\right)$ lie on normal through P, then

$$p^3 . cq - p.\frac{c}{q} = c(p^4 - 1)$$

$$p^3 q^2 - p = p^4 q - q$$

$$p^3 q (p - q) = -(p - q)$$

$$p^3 q = -1$$

(iii) $M(x, y) \Rightarrow M\left(\dfrac{c(p+q)}{2}, \dfrac{\frac{c}{p} + \frac{c}{q}}{2}\right)$

$$= M\left(\frac{c}{2}(p+q), \frac{c}{2}.\frac{(p+q)}{pq}\right)$$

Then, $x = \dfrac{c(p+q)}{2}$ ... (I) and $y = \dfrac{c(p+q)}{2pq}$ ... (II)

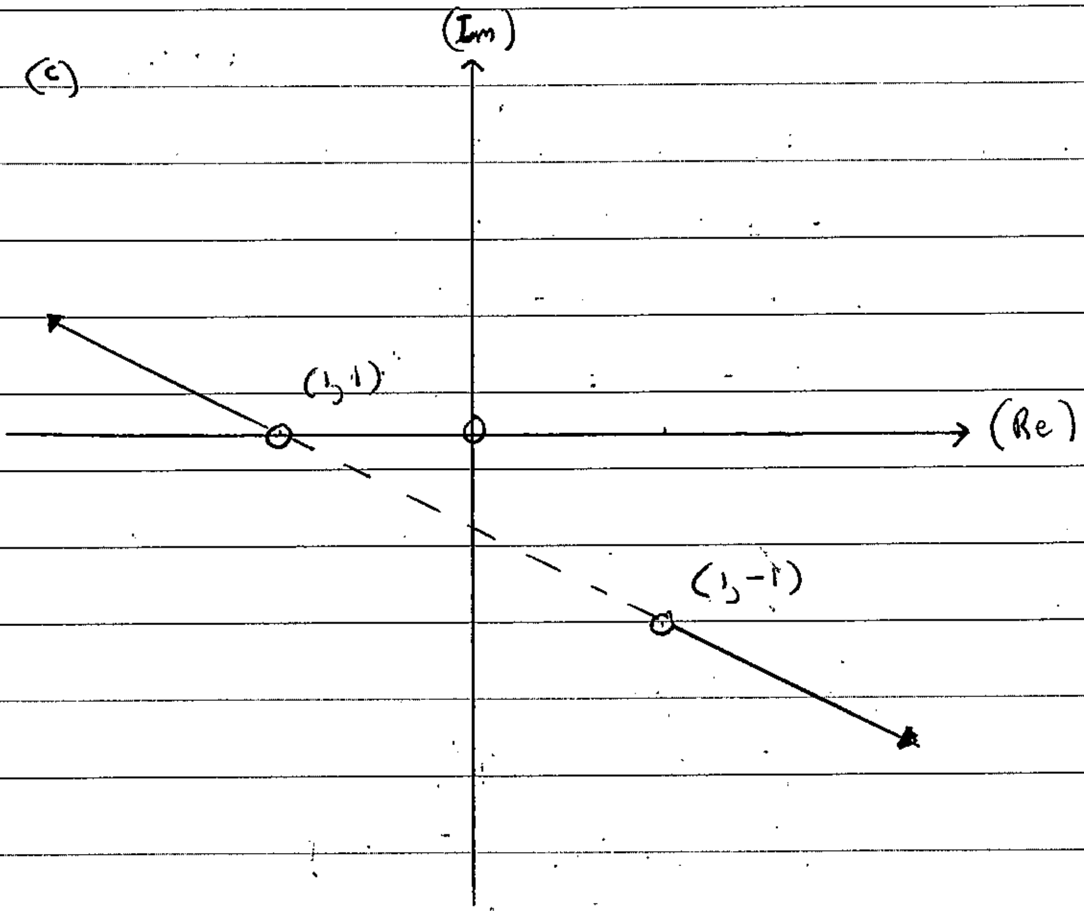
$\therefore$ Substitute (I) into II then $y = \frac{x}{pq}$

Now $p^3 q = -1$ $\Rightarrow$ $\frac{x}{y} = pq$

$pq = -\frac{1}{p^2}$

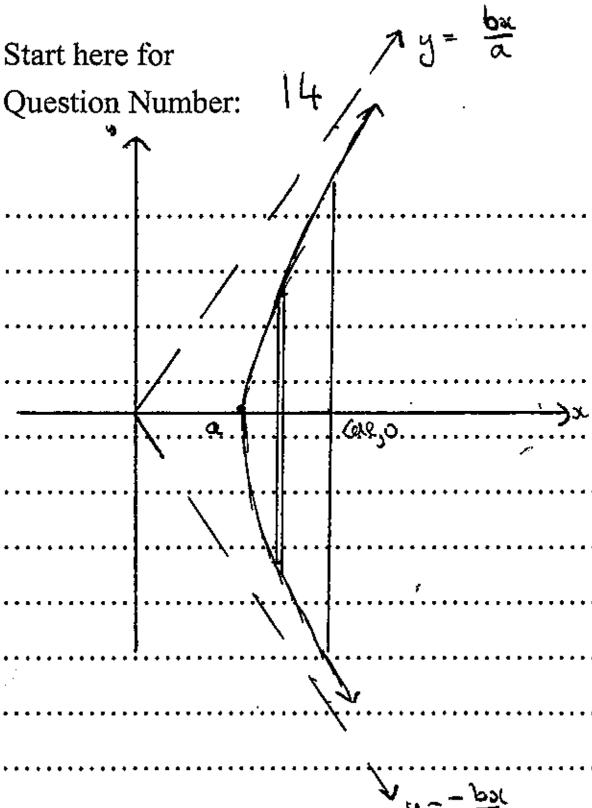
$\therefore \frac{x}{y} = -\frac{1}{p^2}$

(c)

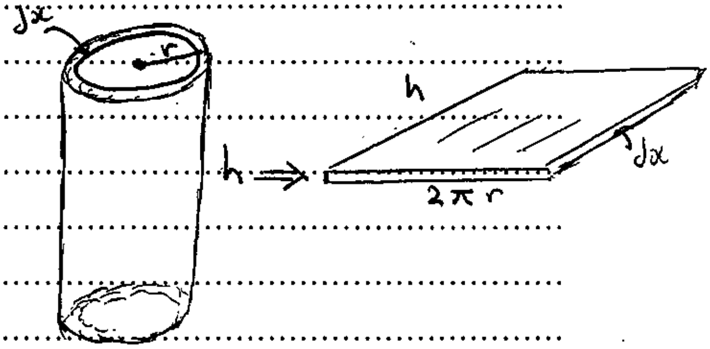


$\arg(z - (-1)) = \arg(z - (1 - i))$

Start here for Question Number: 14



by cylindrical shells



Elemental shell $\delta V = 2\pi r . h . \delta x$

Now $r = x$ and $h = 2y$ $\Big\}$ $\therefore \delta V = 4\pi x y . \delta x$

Given $\dfrac{y^2}{b^2} = \dfrac{x^2}{a^2} - 1$

$\therefore y = \pm \dfrac{b}{a}\sqrt{x^2 - a^2}$

Then $\delta V = 4\pi \, x . \dfrac{b}{a}\sqrt{x^2-a^2} \, \delta x$

$$V = \lim_{\delta x \to 0} \sum_{x=a}^{ae} 4\pi . x . \frac{b}{a}\sqrt{x^2-a^2} . \delta x$$

$$= \frac{4\pi b}{a}\int_a^{ae} \frac{1}{2} . 2x \, (x^2-a^2)^{\frac{1}{2}} . dx$$

$$= \frac{2\pi b}{a}\left[\frac{(x^2-a^2)^{\frac{3}{2}}}{\frac{3}{2}}\right]_a^{ae}$$

$$= \frac{4\pi b}{3a}\left[(a^2e^2-a^2)^{\frac{3}{2}} - (a^2-a^2)^{\frac{3}{2}}\right]$$

$$= \frac{4\pi b}{3a} . a^3 (e^2-1)^{\frac{3}{2}}$$

$* \ \dfrac{b^2}{a^2} = e^2 - 1$

$$= \frac{4\pi a^2 b}{3}\left(\frac{b^2}{a^2}\right)^{\frac{3}{2}}$$

So $V = \dfrac{4\pi a^2 b}{3} \cdot \dfrac{b^3}{a^3}$

$= \dfrac{4\pi b^4}{3a}$

(b) $P(x) = (x^2-3) . Q(x) - 2$
and $P(x) = (x+2) . H(x) + 8$

Now $P(x) = (x-3)(x+2) . B(x) + (ax+b)$

(I) $P(3) = 0 + 3a + b = -2$
$\therefore$ $3a + b = -2$

(II) $P(-2) = 0 + -2a + b = 8$
$\therefore$ $-2a + b = 8$

So (I) − (II) gives $5a = -10$
$a = -2$
on substitution $b = 4$

Remainder is $(-2x + 4)$

(c) (i) DPX lie on a circle since DX subtends right angle at P. [angle in a semi-circle]
Since AD = AX and DX is diameter, A is centre of circle and radius of circle AD = AB, so B lies on circle DPX
DX diameter then subtends right angle at circumference, ie $\angle DBX = 90°$.

(ii) Now, DPBX lie on circle with diameter DX, centre A.
So AP = AB [radii of circle]

Gives $\triangle APB$ is isosceles.

then $\angle APB = \angle ABP$ (angles opposite equal sides are equal)

(d) (I) $\alpha\beta\gamma = -q$

$\therefore \dfrac{\beta\gamma}{\alpha} = \dfrac{-q}{\alpha^2}$

$\dfrac{\alpha\beta}{\gamma} = \dfrac{-q}{\gamma^2}$

$\dfrac{\alpha\gamma}{\beta} = \dfrac{-q}{\beta^2}$

Let $X = \dfrac{-q}{\alpha^2}$

$\therefore \alpha^2 = \dfrac{-q}{X}$

$\alpha = \pm\sqrt{\dfrac{-q}{X}}$

On Substitution

$\left[\pm\left(\sqrt{\dfrac{-q}{X}}\right)\right]^3 - 3p\left[\pm\sqrt{\dfrac{-q}{X}}\right] + q = 0$

$\pm\left(\dfrac{-q}{X}\right)^{\frac{3}{2}} - 3p\left(\pm\left(\dfrac{-q}{X}\right)^{\frac{1}{2}}\right) = -q$

Square both sides

$\left(\dfrac{-q}{X}\right)^3 - 6p\left(\dfrac{-q}{X}\right)^2 + 9p^2\left(\dfrac{-q}{X}\right) = q^2$

$\therefore -q^3 - 6pq^2X - 9p^2qX^2 = q^2X^3$

$\Rightarrow q^2X^3 + 9p^2qX^2 + 6pq^2X + q^3 = 0$

So $X^3 + \dfrac{9p^2X^2}{q} + 6pX + q = 0$

or (II) $\dfrac{-b}{1} = \dfrac{\beta\gamma}{\alpha} + \dfrac{\alpha\gamma}{\beta} + \dfrac{\alpha\beta}{\gamma}$

$= \dfrac{(\beta\gamma)^2 + (\alpha\gamma)^2 + (\alpha\beta)^2}{\alpha\beta\gamma}$

Now $[\beta\gamma + \alpha\gamma + \alpha\beta]^2 = \beta^2\gamma^2 + 2\alpha\beta\gamma^2 + 2\alpha\beta^2\gamma + \alpha^2\gamma^2 + 2\alpha^2\beta\gamma + \alpha^2\beta^2$

$= \beta^2\gamma^2 + \alpha^2\gamma^2 + \alpha^2\beta^2 + 2\alpha\beta\gamma(\alpha+\beta+\gamma)$

So $\beta^2\gamma^2 + \alpha^2\gamma^2 + \alpha^2\beta^2 = [\beta\gamma + \alpha\gamma + \alpha\beta]^2 - 2\alpha\beta\gamma(\alpha+\beta+\gamma)$

$= (-3p)^2 - 2.q.0$

$= 9p^2$

Gives $-b = \dfrac{9p^2}{-q}$

$b = \dfrac{9p^2}{q}$

* $\alpha+\beta+\gamma = 0$

* $\alpha\beta+\alpha\gamma+\beta\gamma = -3p$

* $\alpha\beta\gamma = -q$

* $\dfrac{c}{1} = \dfrac{\beta\gamma}{\alpha}\cdot\dfrac{\alpha\gamma}{\beta} + \dfrac{\beta\gamma}{\alpha}\cdot\dfrac{\alpha\beta}{\gamma} + \dfrac{\alpha\gamma}{\beta}\cdot\dfrac{\alpha\beta}{\gamma}$

$= \gamma^2 + \beta^2 + \alpha^2$

$= (\gamma+\beta+\alpha)^2 - 2(\alpha\beta+\alpha\gamma+\beta\gamma)$

$= 0^2 - 2\times(-3p)$

$= 6p$

* $\dfrac{-d}{1} = \dfrac{\beta\gamma}{\alpha}\cdot\dfrac{\alpha\gamma}{\beta}\cdot\dfrac{\alpha\beta}{\gamma}$

$= \alpha\beta\gamma$

$= -q$

$d = q$

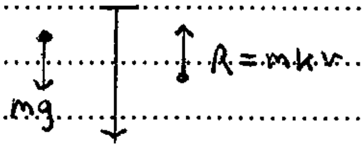
Then $x^3 + bx^2 + cx + d = 0$

$\Rightarrow x^3 + \dfrac{9p^2}{q}x^2 + 6px + q = 0$

Start here for

Question Number: 15

(a) (i)



at $t=0$, $v=0$, $x=0$

$F = mg - mkv$

so $m\ddot{x} = mg - mkv$

$\ddot{x} = g - kv$

Terminal velocity v_T is at $\ddot{x}=0$

So, let $g - kv_T = 0$

$v_T = \frac{g}{k}$

or $\frac{dv}{dt} = g - kv$

$\frac{dt}{dv} = \frac{1}{g-kv}$

$t = -\frac{1}{k}\int \frac{-k}{g-kv}\,dv$

$t = -\frac{1}{k}\Big[\ln(g-kv)\Big]_0^{v_T}$

$= -\frac{1}{k}\Big[\ln(g-kv_T) - \ln(g)\Big]$

$-kt = \ln\left[\frac{g-kv_T}{g}\right]$

$e^{-kt} = \frac{g-kv_T}{g}$

$kv_T = g(1-e^{kt})$

as $t \to \infty$, $e^{kt} \to 0$

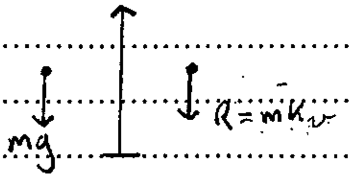
$\therefore kv_T = g$

$v_T = \frac{g}{k}$

(ii) (α) at $t=0$ $\quad m\ddot{x} = -mg - mkv$

$x = 0$ $\quad \ddot{x} = -g - kv$

$v = v_T$



$$\text{Let } \frac{dv}{dt} = -g - kv$$

$$\frac{dt}{dv} = \frac{1}{-g-kv}$$

$$t = -\frac{1}{k}\int_{v_T}^{v} \frac{-k}{-g-kv}\,dv$$

$$-kt = \Big[\ln(-g-kv)\Big]_{v_T}^{v}$$

$$-kt = \ln(-g-kv) - \ln(-g-kv_T)$$

$$= \ln\left[\frac{-(g+kv)}{-(g+kv_T)}\right]$$

$$e^{-kt} = \frac{g+kv}{g+kv_T}$$

So $\quad g e^{-kt} + e^{-kt} k v_T = g + kv$

$$kv = g e^{-kt} + k v_T \cdot e^{-kt} - g$$

$$v = \frac{g}{k} e^{-kt} + v_T \cdot e^{-kt} - \frac{g}{k}$$

$*\ v_T = \dfrac{g}{k}$

$$= v_T \cdot e^{-kt} + v_T e^{-kt} - v_T$$

$$v = v_T\left(2e^{-kt} - 1\right)$$

(β) Let $\quad v \cdot \dfrac{dv}{dx} = -(g+kv)$

$$\frac{dv}{dx} = \frac{-(g+kv)}{v}$$

$$\therefore \quad \frac{dx}{dv} = \frac{-v}{g+kv}$$

Gives $x = -\frac{1}{k}\int_{v_T}^{0} \frac{kv}{g+kv}\,dv$

$= -\frac{1}{k}\int_{v_T}^{0} \frac{g+kv-g}{g+kv}\,dv$

$= -\frac{1}{k}\left[\int_{v_T}^{0} dv - \int_{v_T}^{0}\frac{g}{g+kv}\,dv\right]$

$= -\frac{1}{k}\left[(v)_{v_T}^{0} - \frac{g}{k}\int_{v_T}^{0}\frac{k}{g+kv}\,dv\right]$

$= -\frac{1}{k}\left[(0-v_T) - \frac{g}{k}\left[\ln(g+kv)\right]_{v_T}^{0}\right]$

$= \frac{v_T}{k} + \frac{g}{k^2}\left[\ln(g+0) - \ln(g+kv_T)\right]$

$x = \frac{v_T}{k} - \frac{g}{k^2}\ln\left[\frac{g+kv_T}{g}\right]$

* $v_T = \frac{g}{k}$ So $x = \frac{v_T}{k}\left[1 - \ln\left(1+\frac{v_T}{v_T}\right)\right]$

$= \frac{v_T}{k}\left[1 - \ln 2\right]$

(b) (i) Quadratic Formula gives

$z = \frac{\sin 2\theta \pm \sqrt{(\sin 2\theta)^2 - 4\sin^2\theta}}{2\sin^2\theta}$

$= \frac{2\sin\theta\cos\theta \pm \sqrt{4\sin^2\theta.\cos^2\theta - 4\sin^2\theta}}{2\sin^2\theta}$

$= \cot\theta \pm \frac{2\sin\theta\sqrt{\cos^2\theta - 1}}{2\sin^2\theta}$

$= \cot\theta \pm \frac{\sqrt{i^2(1-\cos^2\theta)}}{\sin\theta}$

$$\therefore \quad z = \cot\theta \pm \frac{i\sin\theta}{\sin\theta} \qquad (1-\cos^2\theta = \sin^2\theta)$$

$$= \cot\theta \pm i$$

<u>So</u> $\alpha = \cot\theta - i \qquad \beta = \cot\theta + i$

$$= \frac{\cos\theta - i\sin\theta}{\sin\theta} \qquad = \frac{\cos\theta + i\sin\theta}{\sin\theta}$$

(ii) $$\alpha^n + \beta^n = \left[\frac{\cos\theta - i\sin\theta}{\sin\theta}\right]^n + \left[\frac{\cos\theta + i\sin\theta}{\sin\theta}\right]^n$$

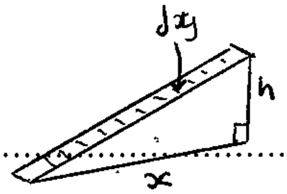
$$= \frac{(\cos\theta - i\sin\theta)^n + (\cos\theta + i\sin\theta)^n}{\sin^n\theta}$$

[by de Moivre] $$= \frac{\cos n\theta - i\sin n\theta + \cos n\theta + i\sin n\theta}{\sin^n\theta}$$

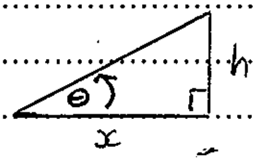
$$= \frac{2\cos n\theta}{\sin^n\theta}$$

Start here for
Question Number: 16

(a) (i) Elemental cross-section slice



Area of 'face'



$\frac{h}{x} = \tan\theta$ AND $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

$h = x.\tan\theta$ ... (II)

$\therefore x^2 = \frac{a^2}{b^2}(b^2-y^2)$ ... (I)

Now, Area $= \frac{1}{2}xh$

$= \frac{x^2.\tan\theta}{2}$ From (I) area $= \frac{\frac{a^2}{b^2}(b^2-y^2).\tan\theta}{2}$

$\therefore$ Area $= \frac{a^2}{2b^2}.(b^2-y^2).\tan\theta$

(ii) Elemental cross section has volume

$\delta V = A.\delta y$

$= \frac{a^2}{2b^2}(b^2-y^2).\tan\theta.\,dy$

So Volume $= \lim_{dy\to 0}\sum_{y=-b}^{b}\frac{a^2}{2b^2}(b^2-y^2).\tan\theta.\,dy$

$= \frac{a^2}{2b^2}\int_{-b}^{b}(b^2-y^2).\tan\theta\,dy$

(* $\tan\theta$ constant)

[even fn] $= \frac{a^2.\tan\theta}{b^2}\int_{0}^{b}(b^2-y^2)\,dy$

Volume $= \frac{a^2.\tan\theta}{b^2}\left[b^2y - \frac{1}{3}y^3\right]_0^b$

$= \frac{a^2.\tan\theta}{b^2}\left[(b^3 - \frac{1}{3}b^3) - 0\right]$

Gives Volume as $\frac{2a^2b\tan\theta}{3}$

(b) (i) (α) $\frac{1}{x} - \frac{1}{x+1} = \frac{(x+1) - x}{x(x+1)}$

$= \frac{1}{x+1}$ $\left[\frac{1!}{x+1}\right]$

(β) $\frac{1}{x} - \frac{2}{x+1} + \frac{1}{x+2} = \frac{(x+1)\cdot(x+2) - 2\cdot x(x+2) + x(x+1)}{x\cdot(x+1)\cdot(x+2)}$

$= \frac{x^2+3x+2 - 2x^2-4x + x^2+x}{x\cdot(x+1)(x+2)}$

$= \frac{2}{x\cdot(x+1)(x+2)}$ $\left(\begin{aligned}2! &= 2\times 1\\ &= 2\end{aligned}\right)$

(ii) L.H.S. $T(k,x) - T(k, x+1)$

$= \frac{k!}{x\cdot(x+1)(x+2)\cdots(x+k)} - \frac{k!}{(x+1)\cdot(x+2)(x+3)\cdots(x+k+1)}$

$= \frac{k!\,(x+k+1) - k!\,x}{x\,(x+1)\,(x+2)\cdots(x+k)(x+k+1)}$

$= \frac{k!\,(k+1)}{x\cdot(x+1)\cdot(x+2)\cdots(x+k+1)}$

$= \frac{(k+1)!}{x\,(x+1)\,(x+2)\cdots(x+k+1)}$

$= T(k+1, x)$ R.H.S.

(iii) For $n=1$, L.H.S $\frac{{}^nC_0}{x} - \frac{{}^nC_1}{x+1} = \frac{1}{x} - \frac{1}{x+1}$

$= \frac{1!}{x(x+1)}$ from (i)

R.H.S. is $\frac{1!}{x(x+1)}$ so true for $n=1$

Let proposition be true for $n=k$ where k is a positive integer >1

Then $T(k,x) \Rightarrow \frac{{}^kC_0}{x} - \frac{{}^kC_1}{x+1} + \frac{{}^kC_2}{x+2} + \cdots + \frac{(-1)^k\,{}^kC_k}{x+k}$

$= \frac{k!}{x(x+1)(x+2)\cdots(x+k)}$

For next $n = k+1$, the L.H.S. is

$$\frac{{}^{k+1}C_0}{x} - \frac{{}^{k+1}C_1}{x+1} + \frac{{}^{k+1}C_2}{x+2} \ldots + (-1)^k \frac{{}^{k+1}C_k}{x+k} + (-1)^{k+1} \frac{{}^{k+1}C_{k+1}}{x+k+1}$$

Now ${}^{k+1}C_0 = {}^kC_0$ and ${}^{k+1}C_{k+1} = {}^kC_k$

Also, ${}^{k+1}C_r = {}^kC_r + {}^kC_{r-1}$

So L.H.S. becomes

$$\frac{{}^kC_0}{x} - \left[\frac{{}^kC_1 + {}^kC_0}{x+1}\right] + \left[\frac{{}^kC_2 + {}^kC_1}{x+2}\right] \ldots + (-1)^k \left[\frac{{}^kC_k + {}^kC_{k-1}}{x+k}\right] + (-1)^{k+1} \frac{{}^kC_k}{x+k+1}$$

$$= \frac{{}^kC_0}{x} - \frac{{}^kC_1}{x+1} + \frac{{}^kC_2}{x+2} + \ldots + \frac{(-1)^k\,{}^kC_k}{x+k}$$
$$- \left[\frac{{}^kC_0}{x+1} - \frac{{}^kC_1}{x+2} + \frac{{}^kC_2}{x+3} + \ldots + (-1)^k \frac{{}^kC_{k-1}}{x+k} + (-1)^{k+1} \frac{{}^kC_k}{x+k+1}\right]$$

$= T(k, x) - T(k, x+1)$

$= T(k+1, x)$ (from (ii))

$= \dfrac{(k+1)!}{x.(x+1)(x+2)\ldots(x+k+1)}$ as required.

If true for $n=k$ it is true for $n=k+1$. So by method of induction true for all k.