Student Number:	
Class:	

TRIAL HIGHER SCHOOL CERTIFICATE EXAMINATION 2015

MATHEMATICS EXTENSION 2

General Instructions:

- Reading Time: 5 minutes.
- Working Time: 3 hours.
- Write in black pen.
- Board approved calculators & templates may be used
- A Standard Integral Sheet is provided.
- In Question 11 - 16, show all relevant mathematical reasoning and/or calculations.
- Marks may not be awarded for careless or badly arranged working.

Total Marks 100

Section I: 10 marks

- Attempt Question 1 – 10.
- Answer on the Multiple Choice answer sheet provided.
- Allow about 15 minutes for this section.

Section II: 90 Marks

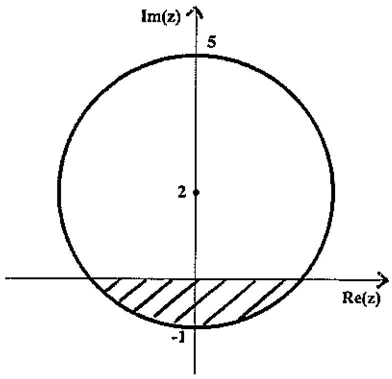
- Attempt Question 11 - 16
- Answer on lined paper provided. Start a new page for each new question.
- Allow about 2 hours & 45 minutes for this section.

The answers to all questions are to be returned in separate *stapled* bundles clearly labelled Question 11, Question 12, etc. Each question must show your Candidate Number.

Section 1 (10 Marks)

Attempt questions 1 – 10. Use the Multiple Choice answer sheet supplied.

1. A circle with centre (0,2) and radius 3 units is shown below on an Argand diagram.



Which of the following inequalities represents the shaded region ?

A) $\text{Im}(z) \leq 0$ and $|z-2| \leq 3$ B) $\text{Re}(z) \leq 0$ and $|z-2| \leq 3$

C) $\text{Im}(z) \leq 0$ and $|z-2i| \leq 3$ D) $\text{Re}(z) \leq 0$ and $|z-2i| \leq 3$

2. The equation $x^2 - xy + y^2 = 3$ defines y implicitly in terms of x.

The expression for $\frac{dy}{dx}$ is :

A) $\frac{y-2x+3}{2y-x}$ B) $\frac{y-2x+3}{x-2y}$ C) $\frac{y-2x}{2y-x}$ D) $\frac{y-2x}{x-2y}$

3. In how many ways can five letters be chosen from the letters ARRANGE ?

A) 9 B) 10 C) 12 D) 21

4. Consider a polynomial $P(x)$ of degree 3.

Two real numbers a and b are such that :

$a < b$

$P(a) > P(b) > 0$

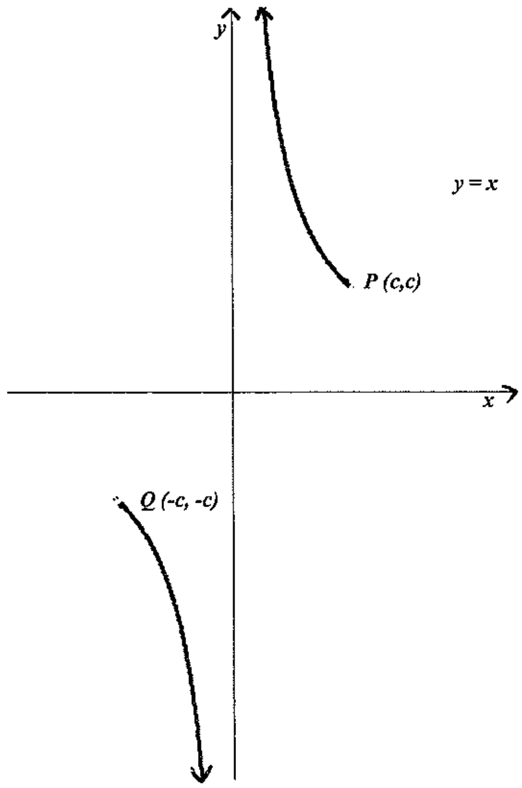
$P'(a) = P'(b) = 0$

The polynomial has :

A) 3 real zeroes B) 1 real zero γ such that $\gamma < a$

C) 1 real zero γ such that $a < \gamma < b$ D) 1 real zero γ such that $\gamma > b$

5. The graph shows a part of the hyperbola $x = ct\,, y = c/t$.



Which pair of parametric equations precisely describes the graph as shown?

A) $x = c(t^2 + 1),\ y = c/(t^2 + 1)$ B) $x = c(1 - t^2),\ y = c/(1 - t^2)$

C) $x = c\sqrt{1 - t^2}\ ,\ y = c/\sqrt{1 - t^2}$ D) $x = c \sin t\ ,\ y = c/\sin t$

6. The region enclosed by $y = x^3$, $y = 0$ and $x = 2$ is rotated about the y-axis to produce a solid. What is the volume of that solid ?

A) $\frac{8\pi}{5}$ units^3 B) $\frac{32\pi}{5}$ units^3 C) $\frac{64\pi}{5}$ units^3 D) $\frac{16\pi}{5}$ units^3

7. The equation $|z - 4| + |z + 4| = 10$ defines an ellipse. What is the length of the semi minor axis ?

A) $2\frac{2}{5}$ B) 3 C) 4 D) 5

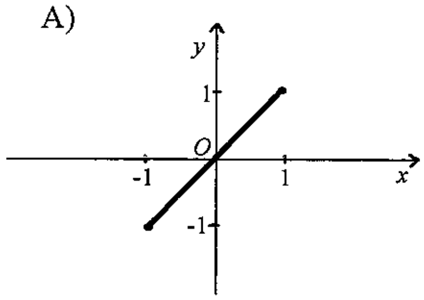
8. A particle is moving in a circle of radius 80cm with a linear speed of 4π m/s.

It has a constant angular speed (in revolutions per minute) of :

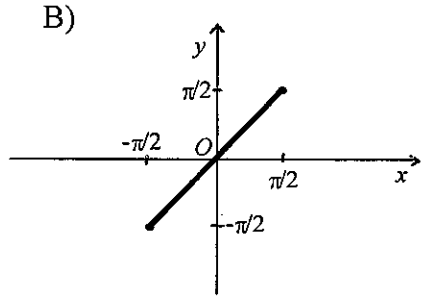
A) 3/8 rpm B) 3/2 rpm C) $37\frac{1}{2}$ rpm D) 150 rpm

9. Which of the diagrams below best represents the graph of $y = \sin^{-1}(\sin x)$?

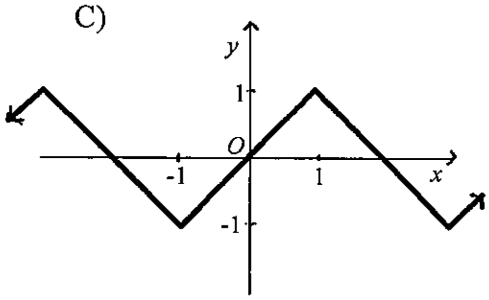
A)



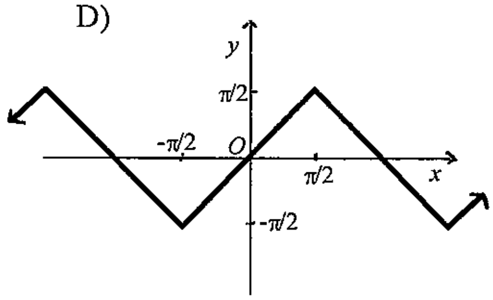
B)



C)



D)



10. A particle of mass m moves in a straight line under the action of a resultant force F where $F=F(x)$. Given that the velocity v is v_0 when the position x is x_0, and that v is v_1 when x is x_1 , it follows that $|v_1| =$

A) $\sqrt{\dfrac{2}{m}}\displaystyle\int_{x_0}^{x_1} \sqrt{F(x)}\, dx + v_0$

B) $\sqrt{2}\displaystyle\int_{\sqrt{x_0}}^{\sqrt{x_1}} F(x)\, dx + v_0$

C) $\sqrt{\dfrac{2}{m}\displaystyle\int_{x_0}^{x_1} \sqrt{F(x)}\, dx + (v_0)^2}$

D) $\sqrt{\dfrac{2}{m}\displaystyle\int_{x_0}^{x_1} \{F(x) + (v_0)^2\}\, dx}$

<u>End of Section 1</u>

Section 2 (90 Marks)

Attempt all questions 11 – 16. Start each question on a new sheet of paper.

QUESTION 11 (Start a new sheet of paper) Marks

a) Find $\int \frac{1}{\sqrt{5+4x-x^2}}dx$ 2

b) i) Find the real numbers a and b such that

$$\frac{3x^2-3x+7}{(x-2)(x^2+9)} \equiv \frac{a}{(x-2)} + \frac{bx+1}{(x^2+9)}$$ 2

ii) Find $\int \frac{3x^2-3x+7}{(x-2)(x^2+9)}dx$ 3

c) i) Prove that $\int_0^a f(x)dx = \int_0^a f(a-x)dx$ 1

ii) Hence find the value of $\int_0^2 x(2-x)^5 dx.$ 2

d) Consider the equation $z^2 + az + (2+i) = 0.$

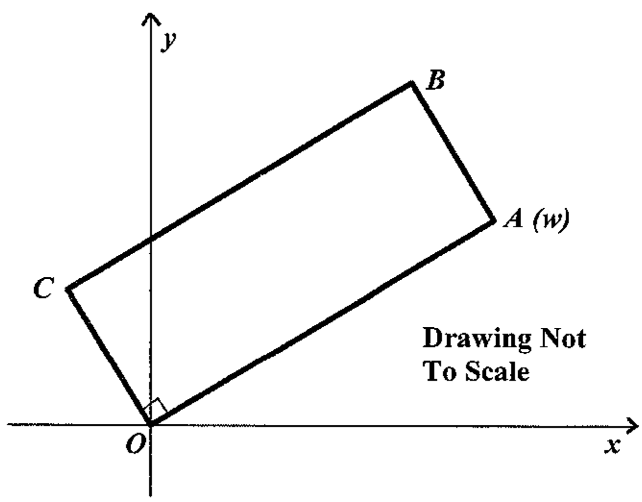
i) Find the complex number a, given that i is a root of the equation. 1

ii) Also write down the second root of the equation. 1

e) In the Argand diagram, $OABC$ is a rectangle, where $3OC = OA$. The vertex at A represents the complex number ω.

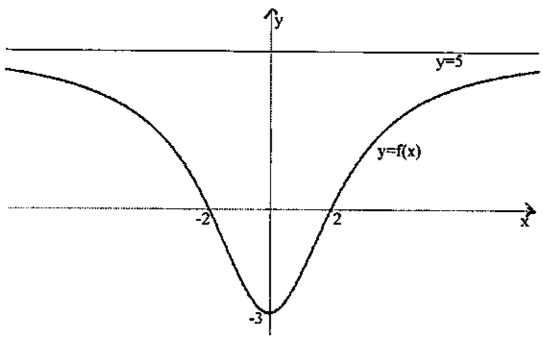
i) What complex number corresponds to the vertex C ? 1

ii) What complex number corresponds to the point of intersection D of the diagonals OB and AC ? 2



QUESTION 12 (Start a new sheet of paper)

a) The diagram below shows the graph of $y = f(x)$.



Draw separate one-third page sketches of the following graphs:

i) $y = \frac{1}{f(x)}$ 2

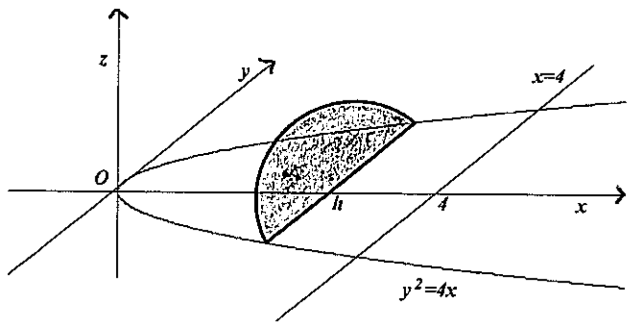
ii) $y^2 = f(x)$ 2

iii) $y = (f(x))^2$ 2

b) i) The polynomial $P(x) = x^4 - 6x^3 + 13x^2 - ax - b$ has two double zeroes. Find a and b. 3

ii) Hence determine the equation of the line which touches the curve $y = x^4 - 6x^3 + 13x^2$ at two distinct points. 1

c) The base of a solid S is the region in the xy plane enclosed by the parabola $y^2 = 4x$ and the line $x = 4$. Each cross section perpendicular to the x axis is a semi-ellipse with the major axis in the xy plane and with the major and minor axes in the ratio a:b.



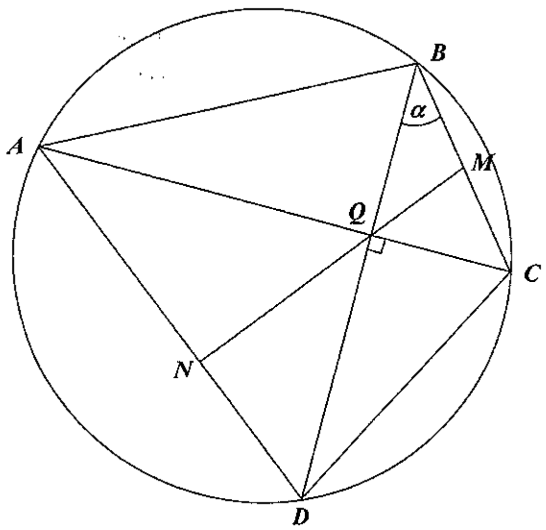
i) Assuming that the area of an ellipse with semi-axes A and B is πAB, show that the area of the semi-ellipse shown at $x = h$ is $2\pi hb/a$. 1

ii) Find the volume of the solid S. 3

iii) The solid T is obtained by rotating the region enclosed by the parabola and the line $x = 4$ about the x axis. Using (with a little care) your result from (ii), or otherwise, find the volume of T. 1

QUESTION 13 (Start a new sheet of paper)

a)



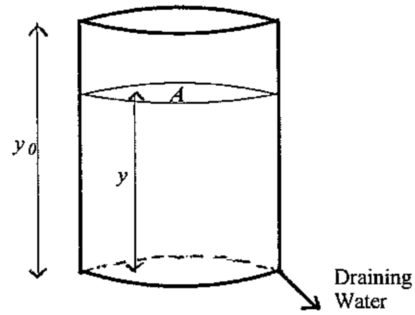
In the diagram above $ABCD$ is a cyclic quadrilateral whose diagonals are perpendicular and intersect at Q. Let M be the midpoint of BC and suppose that MQ produced meets AD at N. Let $\angle QBC = \alpha$.

i) Explain why $BM = QM$. 1

ii) Prove that $MN \perp AD$. 3

b) At a point on a railway line where the radius of curvature is 200m, the track is designed so that a train travelling at 50 km/hr exerts no lateral force on the track. What lateral force would a stationary locomotive, of mass 40 tonnes, exert on the track at this point? (Take g to be 9.81m/s/s.) 4

c) The diagram shows the side view of a vertical cylindrical water cooler of constant cross sectional area A. Water drains through a hole at the bottom of the cooler. It is known that the volume of water decreases at a rate given by $\frac{dV}{dt} = -k\sqrt{y}$ where k is a positive constant and y is the depth of the water. Initially the cooler is full and it would take T seconds to drain completely.



i) Show that $\frac{dy}{dt} = -\frac{k}{A}\sqrt{y}$ 1

ii) Show that $y = y_0\left(1 - \frac{t}{T}\right)^2$ for $0 \le t \le T$. 4

iii) If it takes 10 seconds for half the water to drain, evaluate T. 2

QUESTION 14 (Start a new sheet of paper)

a) A particle moves in Simple Harmonic Motion, the period being 2 seconds and the amplitude 3 metres. Find the maximum speed and the maximum acceleration during the motion. 2

b) i) Use de Moivre's Theorem to show that

$$(\cot\theta + i)^n + (\cot\theta - i)^n = \frac{2\cos n\theta}{\sin^n\theta}$$ 2

ii) Show that the equation $(x+i)^5 + (x-i)^5 = 0$ has roots $0, \pm\cot\frac{\pi}{10}, \pm\cot\frac{3\pi}{10}$. 2

iii) Hence show that the equation $x^4 - 10x^2 + 5 = 0$ has roots $\pm\cot\frac{\pi}{10}, \pm\cot\frac{3\pi}{10}$. 2

iv) Hence show that $\cot\frac{\pi}{10} = \sqrt{5+2\sqrt{5}}$ 2

c) A ball is projected so as to just clear two walls. The first wall is at a height b at a horizontal distance a from the point of projection and the second is of height a at a horizontal distance b from the point of projection. $(b > a)$

It may be assumed that, if the ball is projected from the origin with velocity V at an angle α to the horizontal (x axis), then the equation of the path is given by

$$y = x\tan\alpha - \frac{gx^2\sec^2\alpha}{2V^2}$$

i) Show that the range on the horizontal plane is $\frac{a^2+ab+b^2}{a+b}$. 3

ii) Show that the angle of projection must exceed $\tan^{-1}3$. 2

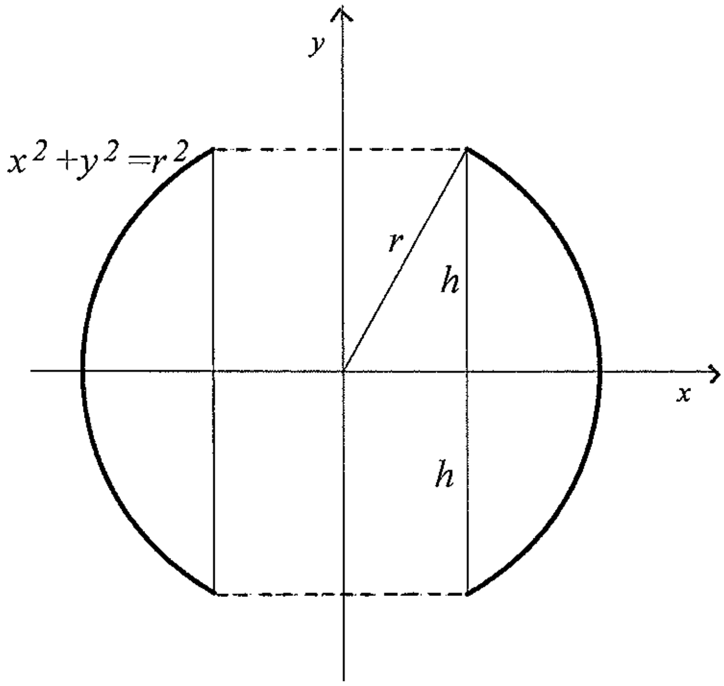
QUESTION 15 (Start a new sheet of paper)

a) The diagram shows a segment of the circle $x^2 + y^2 = r^2$ which is rotated about the y axis to form a collar. This collar is thus a sphere with a symmetrical hole through it. Let the hole be of height $2h$ as shown.

Use the method of cylindrical shells to show that the volume of the material in the collar is given by the integral

$$4\pi \int_{\sqrt{r^2-h^2}}^{r} x\sqrt{r^2 - x^2}\, dx$$

Evaluate the integral to show that the volume of material in the collar is a function of h only and independent of r. 4



b) Consider the equation $z^5 + 32 = 0$.

i) Write down the roots of this equation in modulus argument form. 2

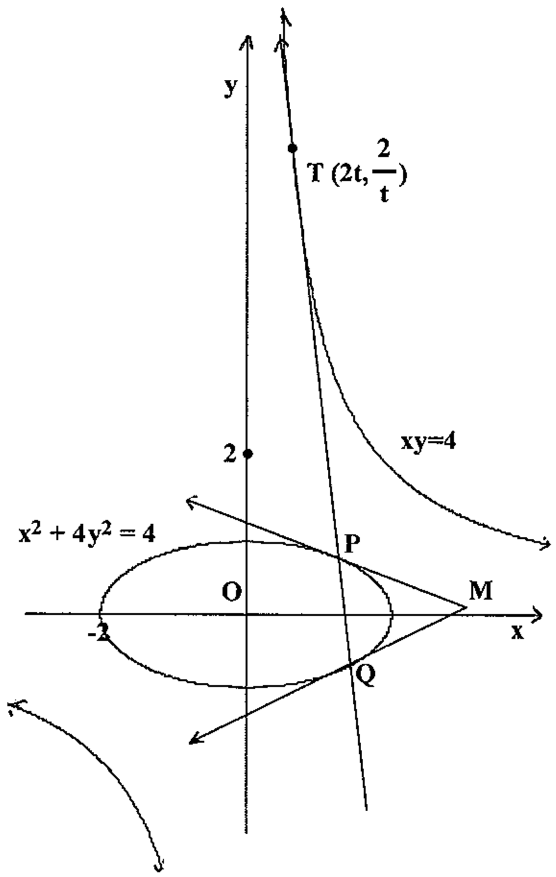
ii) Illustrate these roots on an Argand diagram. 1

iii) If the points A, B, C, D and E cyclically represent these roots, find the area of triangle ACD. (Give your answer to 2 decimal places) 3

Question 15 is continued on the next page ....

c) i) Consider the diagram below. The tangent at $T\,(2t\,,\,2/t)$ to the rectangular hyperbola $xy = 4$ meets the ellipse $x^2 + 4y^2 = 4$ at P and Q. The tangents to the ellipse at P and Q intersect at M. Find the equation of the locus of M. (You may use standard forms of tangents and such things without proof.) 3

ii) Describe briefly the locus and any restrictions that it may have. 2



QUESTION 16 (Start a new sheet of paper)

a) A box contains n jellybeans, some white and some black. Alan and Betty take turns picking a jellybean from the box, without looking, until the box is empty. Alan picks first.

i) If there is 1 black and $n-1$ white jellybeans and n is odd, find the probability that Alan picks the black jellybean. 1

ii) If there are 2 black and $n-2$ white jellybeans and n is even, find the probability that Alan is the first to pick a black jellybean. 2

iii) If there are 2 black and $n-2$ white jellybeans and n is odd , find the probability that Alan is the first to pick a black jellybean. 2

Question 16 is continued on the next page....

Question 16 (continued)

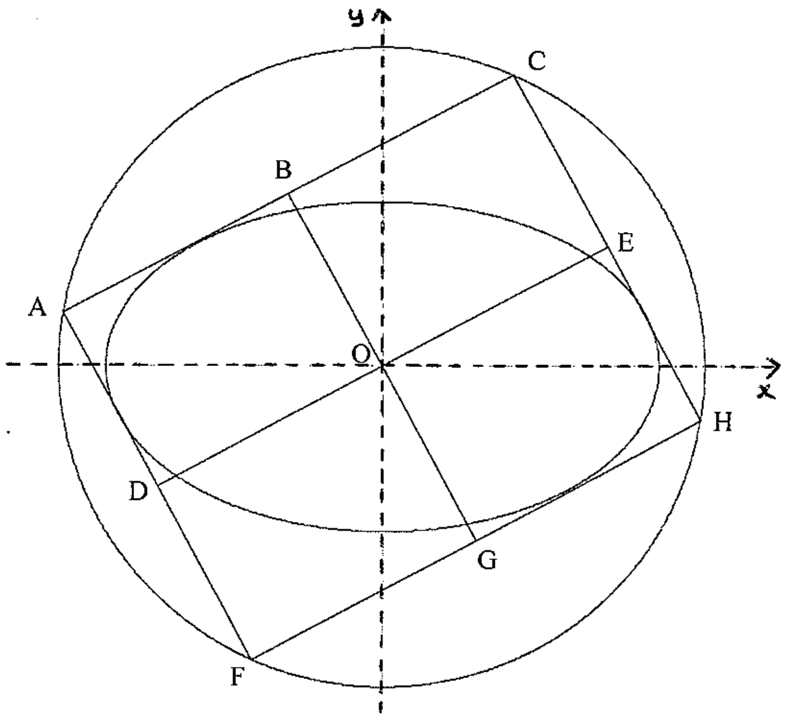
b) Consider the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

i) Show that lines with equations $y = mx \pm \sqrt{a^2m^2 + b^2}$ are always tangent to the ellipse for any real value of m. 2

ii) From any point external to the ellipse, two tangents may be drawn. By considering the above as a quadratic equation in m, or otherwise, show that the locus of points where the two tangents are perpendicular to each other is a circle with equation $x^2 + y^2 = a^2 + b^2$. 2

iii) Show that the area of the ellipse's circumscribing rectangle of which $y = mx \pm \sqrt{a^2m^2 + b^2}$ are two parallel sides, is given by

$$A = \frac{4}{(1+m^2)}\sqrt{(a^2 + m^2b^2)(a^2m^2 + b^2)}$$ 3



You are given that this formula for the area can be rearranged to give the following form for the square of the area: $A^2 = 16a^2b^2 + \frac{16(a^2 - b^2)^2}{(m + \frac{1}{m})^2}$

iv) Show that, for any $m > 0, m + \frac{1}{m} \geq 2$. 1

v) Hence, or otherwise, find the maximum and minimum areas of rectangles which circumscribe the ellipse. 2

END OF EXAM

JRAHS EXTENSION 2 MATHS TRIAL 2015 (SOLUTIONS)

1/ B
2/ C
3/ C
4/ B
5/ D
6/ C
7/ B
8/ D
9/ D
10/ C

2/ $2x - y - x\frac{dy}{dx} + 2y\frac{dy}{dx} = 0$

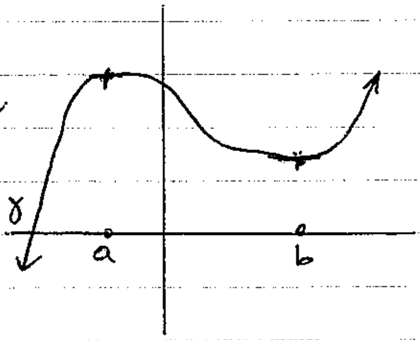
$\frac{dy}{dx}(2y - x) = y - 2x$

$\frac{dy}{dx} = \frac{y-2x}{2y-x}$

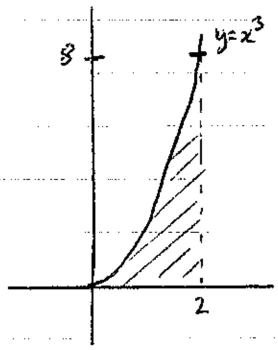
3/

2R	2A	3	ways
1R	2A	3	"
2R	1A	3	"
1R	1A	1	"
0R	2A	1	"
2R	0A	1	"
		12 ways	

4/



6/



$\pi . 2^2 . 8 - \int_0^8 \pi x^2 \, dy$

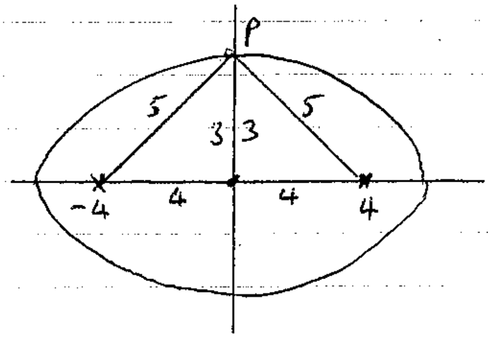
$= \pi\left(32 - \int_0^8 y^{2/3} dy\right)$

$= \pi\left(32 - \left[\frac{3y^{5/3}}{5}\right]_0^8\right)$

$= \pi\left(32 - 32 \times \frac{3}{5}\right)$

$= 64\pi/5$

7/



8/ $0.8\,\omega = 4\pi$ (m/sec)

$\omega = 4\pi/0.8 = 5\pi$ (rad/sec)

$= 300\pi$ rad/min

$= 150$ rev/min.

10/ $F(x) = m\ddot{x}$

$= m\frac{d}{dx}\left(\frac{v^2}{2}\right)$

$\int_{x_0}^{x_1} F(x)\,dx = m\left[\frac{v^2}{2}\right]_{v_0}^{v_1}$

$\frac{2}{m}\int_{x_0}^{x_1} F(x)\,dx = v_1^2 - v_0^2$

$v_1^2 = \frac{2}{m}\int_{x_0}^{x} F(x)\,dx + v_0^2$

$|v_1| = \sqrt{\frac{2}{m}\int_{x_0}^{x_1} F(x)\,dx + v_0^2}$

Question 11

a) $\displaystyle\int \frac{dx}{\sqrt{9-(x-2)^2}} = \sin^{-1}\frac{(x-2)}{3} + k$

b) i) $a(x^2+9)+(bx+1)(x-2)$
$\equiv 3x^2-3x+7$

Set $x=2$

$13a = 13 \qquad a=1$

Coeff of x^2: $a+b=3 \quad \therefore b=2$

(ii) $\displaystyle\int \frac{3x^2-3x+7}{(x-2)(x^2+9)} = \int \frac{dx}{(x-2)} + \int \frac{2x+1}{x^2+9}dx$

$\displaystyle = \int \frac{1}{(x-2)}dx + \int \frac{2x}{x^2+9}dx + \int \frac{dx}{x^2+9}$

$= \ln|x-2| + \ln(x^2+9) + \frac{1}{3}\tan^{-1}\left(\frac{x}{3}\right) + k$

c) Let $I = \displaystyle\int_0^a f(x)\,dx$

Set $u = a-x \longrightarrow x = a-u$

"$du = -dx$"

When $x=a,\ u=0$

$x=0,\ u=a$

$\therefore I = \displaystyle\int_a^0 f(a-u)\cdot -du$

$= -\displaystyle\int f(a-u)\,du$

$= \displaystyle\int_0^a f(a-u)\,du$

$= \displaystyle\int_0^a f(a-x)\,dx$

(Dummy variable)

ii) $I = \displaystyle\int_0^2 x(2-x)^5\,dx$

$= \displaystyle\int_0^2 (2-x)x^5\,dx$ (Using (i))

$= \displaystyle\int_0^2 2x^5 - x^6\,dx$

$= \left[\frac{x^6}{3} - \frac{x^7}{7}\right]_0^2 = \frac{64}{3} - \frac{128}{7}$

$= \frac{7\times 64 - 3\times 128}{21} = \frac{64}{21}$

d) i) By factor thm:
$(i)^2 + a(i) + 2 + i = 0$

$i(a+1) = -1$

$a+1 = -1/i = i$

$a = i-1$

ii) Let other root be β

Sum of roots $= -a = \beta + i$

$\therefore (1-i) = \beta + i$

$\beta = (1-i) - i$

$= 1-2i$

e) i) Rotation $\equiv$ Multiply by i
The length is shortened by a factor of 3.

$\therefore C$ represents $i\omega/3$

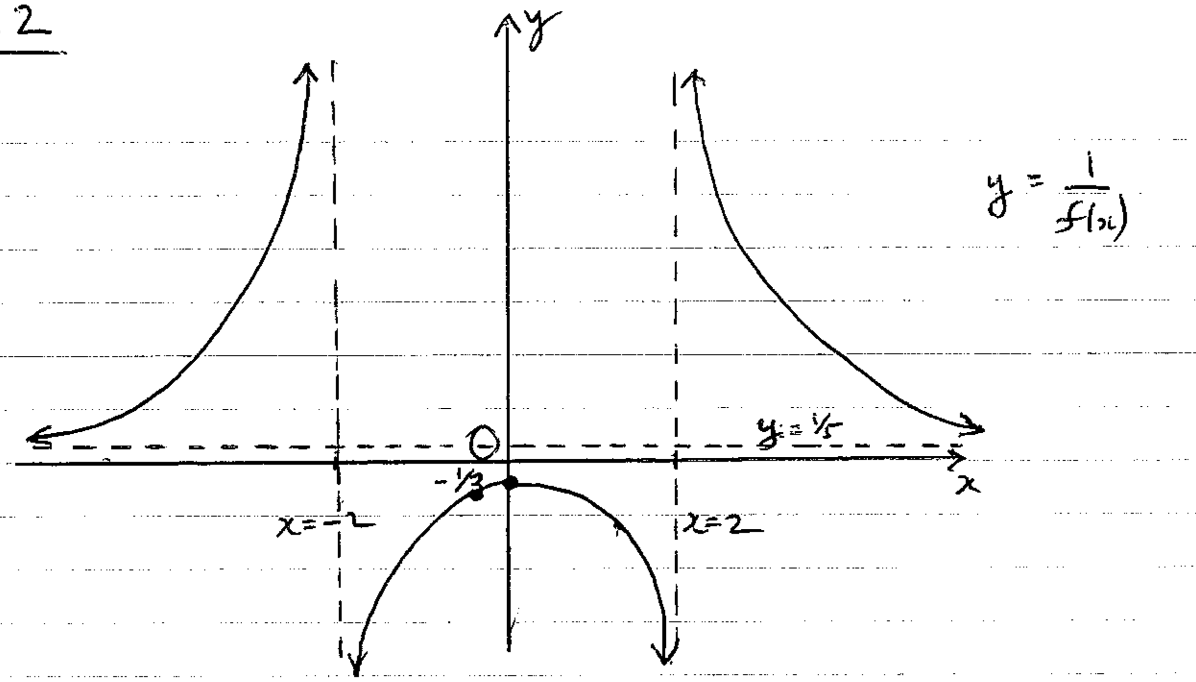
ii) Diagonals bisect each other.
$|OD| = \frac{1}{2}|OB|$

$OB = OA + AB = OA + OC$

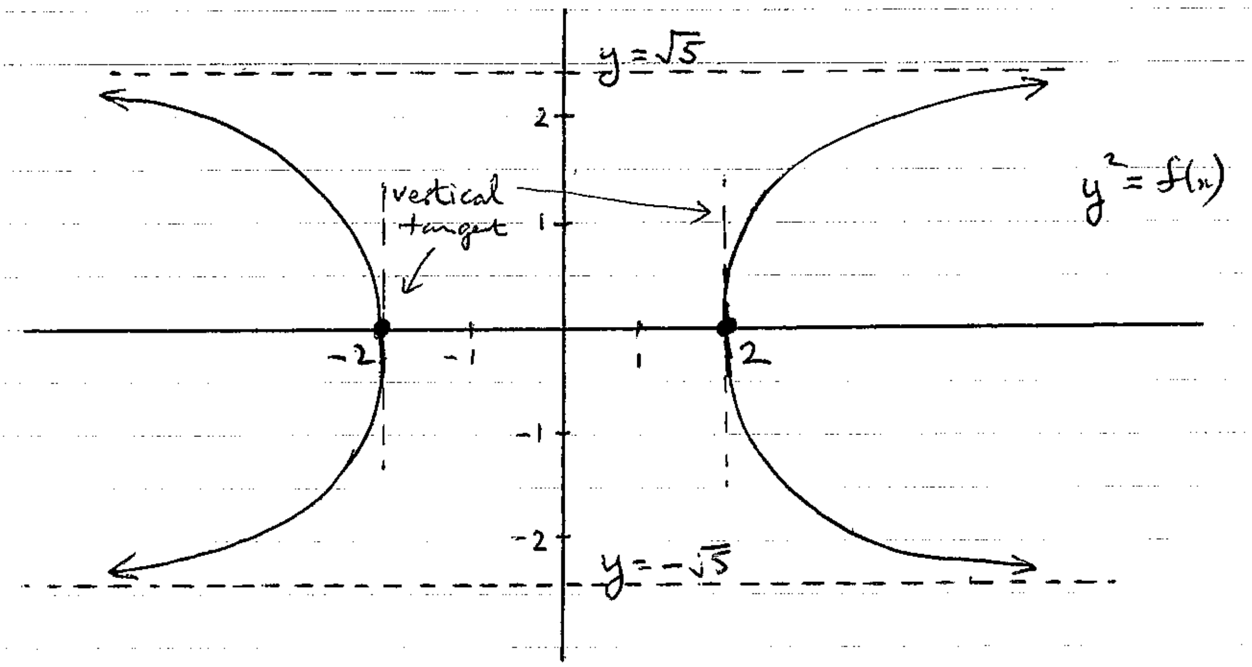
$OD = \frac{1}{2}\left(\omega + \frac{i\omega}{3}\right) = \omega\left(\frac{1}{2} + \frac{i}{6}\right)$

Question 12

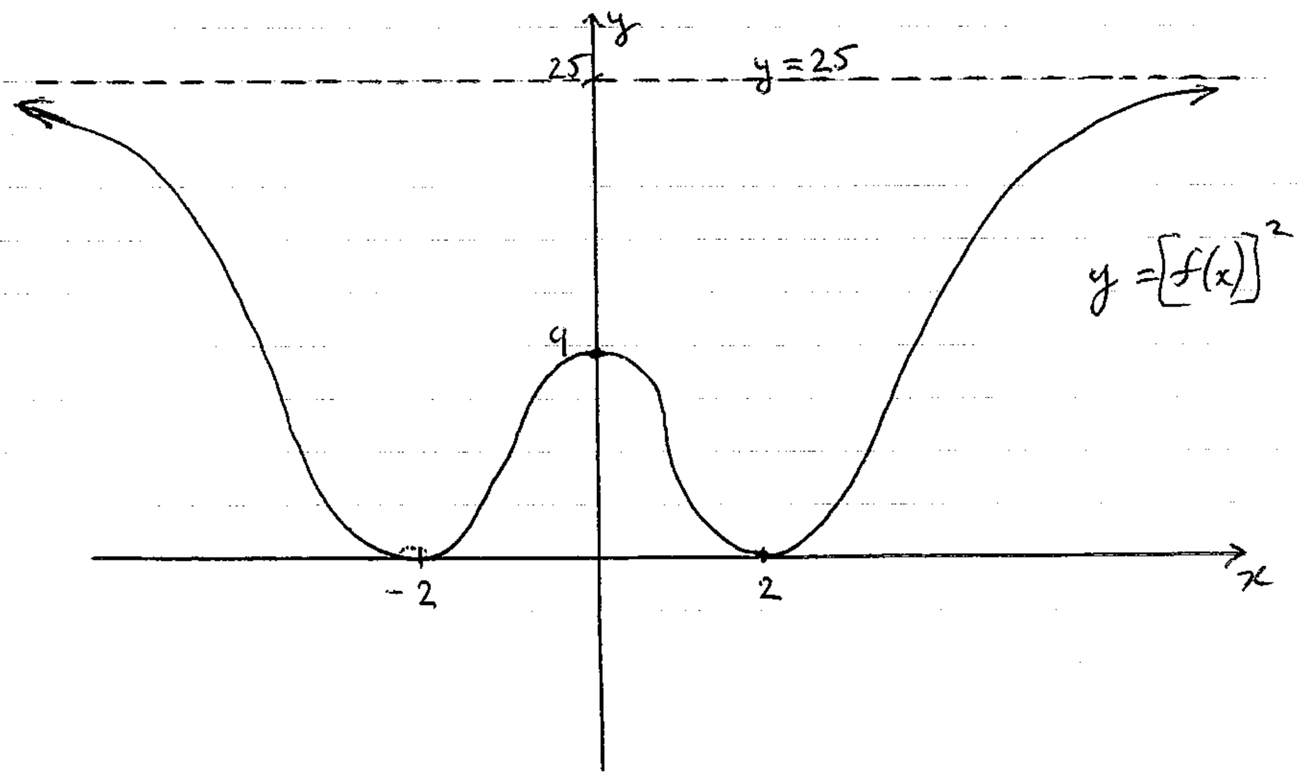
a) i)



ii)



iii)



Question 12 (cont)

b) i) Let the zeros of $P(x)$ be $\alpha, \alpha, \beta, \beta$.

$\therefore\ 2(\alpha+\beta) = 6$ (sum of roots)
$\underline{\alpha+\beta = 3}$ (1)

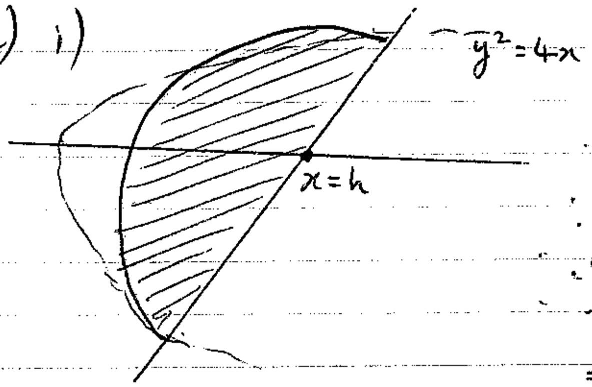
$\alpha^2+\beta^2+4\alpha\beta = 13$ (sum of roots 2 at a time)
$(\alpha+\beta)^2+2\alpha\beta = 13$
$\therefore\ 2\alpha\beta = 4$ (Using (1) above)
$\underline{\alpha\beta = 2}$

Now $b = -\alpha^2\beta^2$ (Product of roots)
$= -(\alpha\beta)^2$
$\underline{b = -4}$

$a = 2\alpha^2\beta + 2\beta^2\alpha$ (Sum of roots 3 at a time)
$= 2\alpha\beta(\alpha+\beta)$
$\underline{a = 12}$

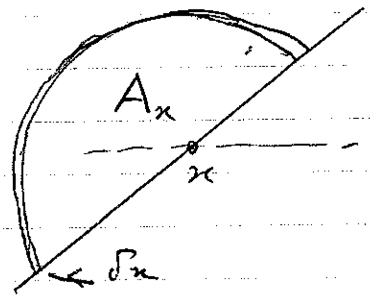
ii) Two touching points is the above scenario if we solve $x^4-6x^3+13x^2$ against $ax+b$
$\therefore$ Line is $\underline{y = 12x-4}$

c) i)



If $x=h$, $y = \pm 2\sqrt{h}$
$\therefore$ Length of major axis is $4\sqrt{h}$.
$\therefore$ Length of minor axis is $4b\sqrt{h}/a$
$\therefore$ Semi Major, Semi Minor are $2\sqrt{h}$, $2b\sqrt{h}/a$
$\therefore$ Area of Full Ellipse $= 4\pi hb/a$
$\underline{\text{Area of Half Ellipse shown} = 2\pi hb/a}$

ii) $\therefore\ \delta V \doteqdot \dfrac{2\pi x b}{a}\,\delta x$



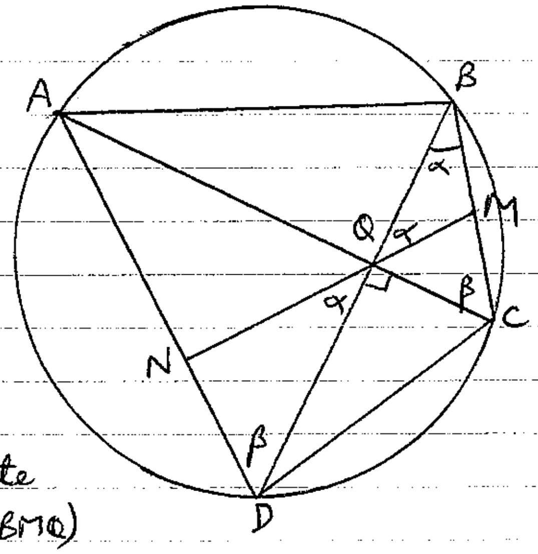
$$V = \lim_{\delta x \to 0} \sum_{x=0}^{4} \frac{2\pi x b}{a}\,\delta x$$
$$= \frac{2\pi b}{a}\int_0^4 x\,dx = \frac{2\pi b}{a}\left[\frac{x^2}{2}\right]_0^4$$
$$= \underline{16\pi b/a \text{ unit}^3}$$

iii) This will generate circles rather than semi ellipses
Hence ratio $b/a = 1$. But it will include two halves

$\therefore\ V\Omega = \underline{32\pi \text{ unit}^3}$

Question 13

a) i) $\angle BQC$ is right angle
(Given that diagonals perpendicular)
$\therefore$ ΔBQC is an angle in a semicircle with BC as diameter and M as the circle centre.
MB and QM are radii of this circle
Thus equal. i.e. $\underline{BM = QM}$

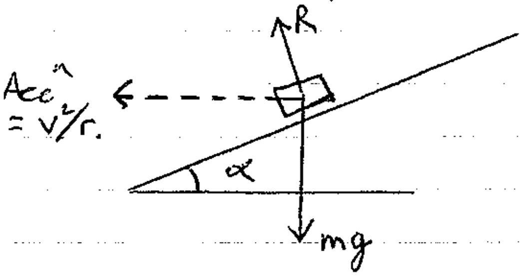


ii) $\therefore \angle BQM = \alpha$ (Equal angles opposite equal sides in ΔBMQ)
$\therefore \angle NQD = \alpha$ (Vertically opposite angles are equal

Now let $\angle ACB = \beta$
Then $\angle ADQ = \beta$ (Angles standing on the same arc are equal

In ΔBQC, $\alpha + \beta + 90^\circ = 180^\circ$
In ΔQND, $\alpha + \beta + \angle QND = 180^\circ$ (Angle sum of triangle is 180°)
$\therefore \angle QND = 90^\circ$
i.e. $\underline{MN \perp AD}$

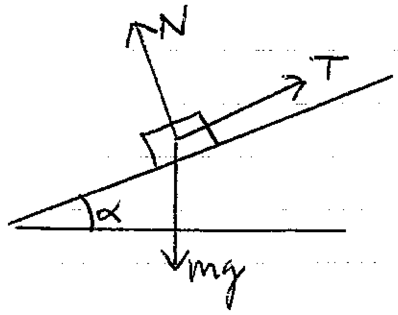
b) Let angle of rail inclination be α. There is no lateral force at the design speed, 50 km/hr



$50\text{ k/h} = \frac{50000}{3600}\text{ m/s} = 13\tfrac{8}{9}\text{ m/s}$

Resolving vertically $R\cos\alpha = mg$ ①
horizontally $R\sin\alpha = mv^2/r$ ②

② ÷ ① $\tan\alpha = \frac{v^2}{rg} = \frac{(13\frac{8}{9})^2}{200 \times 9.81}$ $(= 0.0983$



No movement or acceleration
Resolve vertically $N\cos\alpha + T\sin\alpha = mg$
horizontally $N\sin\alpha - T\cos\alpha = 0$
$\therefore N = T\cot\alpha$

$\therefore \frac{T\cos^2\alpha}{\sin\alpha} + T\sin\alpha = mg$

$T(\cos^2\alpha + \sin^2\alpha) = mg\sin\alpha$

$\therefore$ With $m = 40000$, $\sin\alpha = 0.09785$

$T = 38396\text{N} = 38.4\text{ kN}$ (to 3SF)

Question 13 (cont)

c) i) $V = Ay$

$\therefore \dfrac{dV}{dt} = A\dfrac{dy}{dt}$

$\therefore \dfrac{dy}{dt} = \dfrac{1}{A}\dfrac{dV}{dt} = \dfrac{1}{A}\left(-k\sqrt{y}\right) = \underline{\underline{\dfrac{-k\sqrt{y}}{A}}}$

ii) $\therefore \dfrac{dt}{dy} = \dfrac{-A}{k}y^{-1/2}$

$t = -\dfrac{2Ay^{1/2}}{k} + c$

When $t=T$, $y=0$ $\therefore c = T$

$\therefore t = -\dfrac{2A\sqrt{y}}{k} + T$ $\ast$

When $t=0$, $y=y_0$

$\therefore T = \dfrac{2A\sqrt{y_0}}{k} \Longrightarrow \dfrac{A}{k} = \dfrac{T}{2\sqrt{y_0}}$

Sub this into $\ast$

$t = -\dfrac{2T}{2\sqrt{y_0}}\sqrt{y} + T$

$\dfrac{t}{T} = 1 - \dfrac{\sqrt{y}}{\sqrt{y_0}}$

$\sqrt{\dfrac{y}{y_0}} = 1 - \dfrac{t}{T}$

$\dfrac{y}{y_0} = \left(1 - \dfrac{t}{T}\right)^2$

$\underline{\underline{y = y_0\left(1 - \dfrac{t}{T}\right)^2}}$

iii) $t = 10$ when $y = y_0/2$, $\therefore \dfrac{1}{2} = \left(1 - \dfrac{10}{T}\right)^2 \Rightarrow 1 - \dfrac{10}{T} = \dfrac{1}{\sqrt{2}}$

$\Rightarrow \dfrac{10}{T} = 1 - \dfrac{1}{\sqrt{2}} = \dfrac{\sqrt{2}-1}{\sqrt{2}} \Rightarrow T = \dfrac{10\sqrt{2}}{\sqrt{2}-1}$ secs $= 10\sqrt{2}(\sqrt{2}+1)$ secs.

$= \underline{20 + 10\sqrt{2}}$ secs.

Question 14 (cont)

iii) $(x+i)^5+(x-i)^5=0$

$$\left(x^5+{}^5C_1x^4(i)+{}^5C_2x^3(i)^2+{}^5C_3x^2(i)^3+{}^5C_4x(i)^4+i^5\right)$$
$$+\left(x^5+{}^5C_1x^4(-i)+{}^5C_2x^3(-i)^2+{}^5C_3x^2(-i)^3+{}^5C_4x(-i)^4+(-i)^5\right)=0$$

The imaginary terms cancel

$$2x^5+2\,{}^5C_2x^3i^2+2\,{}^5C_4x\,i^4=0$$

But ${}^5C_2=10$, ${}^5C_4=5$, $i^2=-1$. $\therefore$ $x^5-10x^3+5x=0$ is the equivalent equation

$$x(x^4-10x^2+5)=0$$

But the 0 root is given by x factor.

$\therefore$ roots of $x^4-10x^2+5=0$ are $\pm\cot\pi/10, \pm\cot 3\pi/10$

iv) Solve: $x^2=\dfrac{10\pm\sqrt{100-20}}{2}=5\pm2\sqrt5$

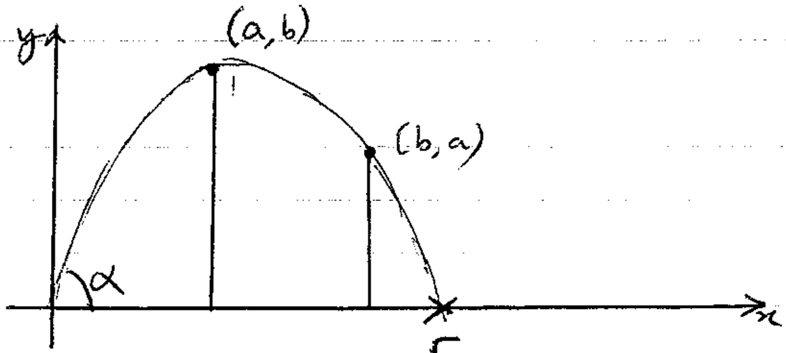
$\therefore$ Four roots are $\pm\sqrt{5\pm2\sqrt5}$.

But the positive roots are $\cot\pi/10$, $\cot 3\pi/10$

The largest positive root is $\cot\pi/10$.

$\therefore$ $\cot\pi/10=\sqrt{5+2\sqrt5}$

c)



The points (a,b) and (b,a) lie on the path

$\therefore$ $b=a\tan\alpha-\dfrac{ga^2\sec^2\alpha}{2V^2}$ ①

$a=b\tan\alpha-\dfrac{gb^2\sec^2\alpha}{2V^2}$ ②

to be solved for V and α.

Question 14

a) $\ddot{x} = -n^2 x \quad \left(2 = \frac{2\pi}{n} \to n = \pi\right)$

$\therefore \frac{d}{dx}\left(\frac{v^2}{2}\right) = -\pi^2 x$

$\frac{v^2}{2} = -\frac{\pi^2 x^2}{2} + k$

When $x = 3$, $v = 0$ $\therefore k = 9\pi^2/2$

$\therefore v^2 = (9 - x^2)\pi^2$

Max speed when $x = 0$ $\quad \underline{\underline{|v|_{max} = 3\pi \text{ m/sec}}}$

Max accⁿ at $x = -3$ $\quad \underline{\underline{\ddot{x}_{max} = 3\pi^2 \text{ m/sec}^2}}$

b) i) $(\cot\theta + i)^n + (\cot\theta - i)^n = \left(\frac{\cos\theta + i\sin\theta}{\sin\theta}\right)^n + \left(\frac{\cos\theta - i\sin\theta}{\sin\theta}\right)^n$

$= \frac{1}{\sin^n\theta}\left\{(\cos\theta + i\sin\theta)^n + (\cos(-\theta) + i\sin(-\theta))^n\right.$

(de Moivre) $= \frac{1}{\sin^n\theta}\left(\cos n\theta + i\sin n\theta + \cos(-n\theta) + i\sin(-n\theta)\right)$

$= \frac{1}{\sin^n\theta}\left(\cos n\theta + i\sin n\theta + \cos n\theta - i\sin n\theta\right)$

$= \underline{\underline{\frac{2\cos n\theta}{\sin^n\theta}}}$

ii) Put $n = 5$, $x = \cot\theta$ where θ are roots of $\frac{2\cos 5\theta}{\sin^5\theta} = 0$

i.e. $\cos 5\theta = 0$

$5\theta = 2k\pi \pm \pi/2$ (general solution for cos)

$= \pm\frac{\pi}{2}$ $(k=0)$,

$2\pi \pm \pi/2$ $(k=1)$ or $-2\pi \pm \pi/2$ $(k=-1)$

$\therefore \theta = \pm\pi/10,\ 5\pi/10,\ 3\pi/10,\ -5\pi/10,\ -3\pi/10.$

The roots of equation are $\cot(\pi/2)$, $\cot(\pm\pi/10)$, $\cot(\pm 3\pi/10)$

($\cot(-\pi/2)$ is same as $\cot(\pi/2)$)

i.e. since $\cot(-\theta) = \cot\theta$, roots are $\underline{0, \pm\cot\pi/10, \pm\cot 3\pi/10}$

$\cot(\pi/2)$

Question 14 (cont)

① $\times b^2$ $\quad b^3 = ab^2\tan\alpha - \dfrac{ga^2b^2\sec^2\alpha}{2V^2}$ ③

② $\times a^2$ $\quad a^3 = a^2b\tan\alpha - \dfrac{ga^2b^2\sec^2\alpha}{2V^2}$ ④

③ - ④ $\quad b^3 - a^3 = \tan\alpha \; ab(b-a)$

$$\therefore \tan\alpha = \frac{b^3-a^3}{ab(b-a)} = \underline{\underline{\frac{b^2+ab+a^2}{ab}}} \quad *$$

① $\times b$ $\quad b^2 = ab\tan\alpha - \dfrac{ga^2b\sec^2\alpha}{2V^2}$ ⑤

② $\times a$ $\quad a^2 = ab\tan\alpha - \dfrac{gab^2\sec^2\alpha}{2V^2}$ ⑥

⑤ - ⑥ $\quad b^2 - a^2 = \dfrac{g\sec^2\alpha}{2V^2} \cdot ab(b-a)$

$$\underline{\underline{\frac{2V^2}{g\sec^2\alpha} = \frac{ab}{a+b}}}$$

If the range is r, then (r, 0) lies on the curve

$$\therefore 0 = r\tan\alpha - \frac{gr^2\sec^2\alpha}{2V^2}$$

Rejecting the r = 0, solution

$$\frac{gr\sec^2\alpha}{2V^2} = \tan\alpha$$

$$r = \tan\alpha \; \frac{2V^2}{g\sec^2\alpha} = \frac{b^2+ab+a^2}{ab} \; \frac{ab}{a+b} = \underline{\underline{\frac{b^2+ab+a^2}{a+b}}}$$

ii) From * $\quad \tan\alpha = \dfrac{b^2+ab+a^2}{ab}$

But since $(a-b)^2 > 0$ (since $b \neq a$, given)

$$a^2+b^2-2ab > 0$$

$$a^2+b^2 > 2ab$$

$$\therefore \tan\alpha > \frac{2ab+ab}{ab}.$$

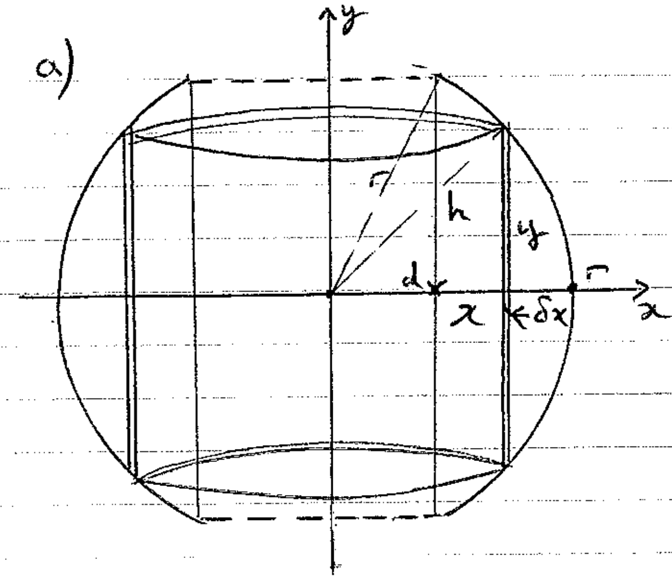
$$\tan\alpha > 3$$

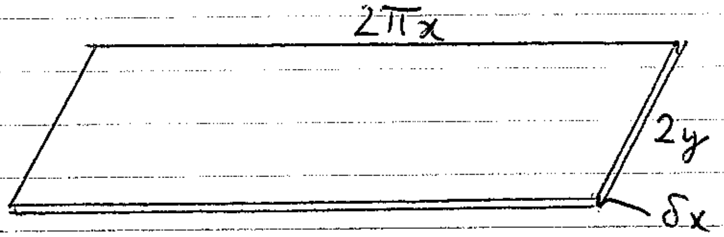
$$\underline{\underline{\alpha > \tan^{-1}(3)}}$$

Since α is acute and $\tan^{-1}$ is an increasing function.

Question 15

a)





$$\delta V \doteqdot 4\pi x y\, \delta x$$

$$\therefore V \doteqdot \lim_{\delta x \to 0} \sum_{x=d}^{r} 4\pi x y\, \delta x$$ where d as in diagram

$$= 4\pi \int_d^r xy\, dx$$

But, by Pythagoras, $d = \sqrt{r^2 - h^2}$, $y = \sqrt{r^2 - x^2}$

$$\therefore V = 4\pi \int_{\sqrt{r^2-h^2}}^{r} x\sqrt{r^2 - x^2}\, dx$$

$$= \left[(4\pi)\left(\frac{-1}{3}\right)(r^2 - x^2)^{3/2} \right]_{\sqrt{r^2-h^2}}^{r}$$

$$= \frac{4\pi}{3}\left\{ -0 + \left(r^2 - (r^2 - h^2)\right)^{3/2} \right\} = \frac{4\pi h^3}{3}$$ (independent of r)

b) i) $z^5 = 2^5(-1)$ Roots are

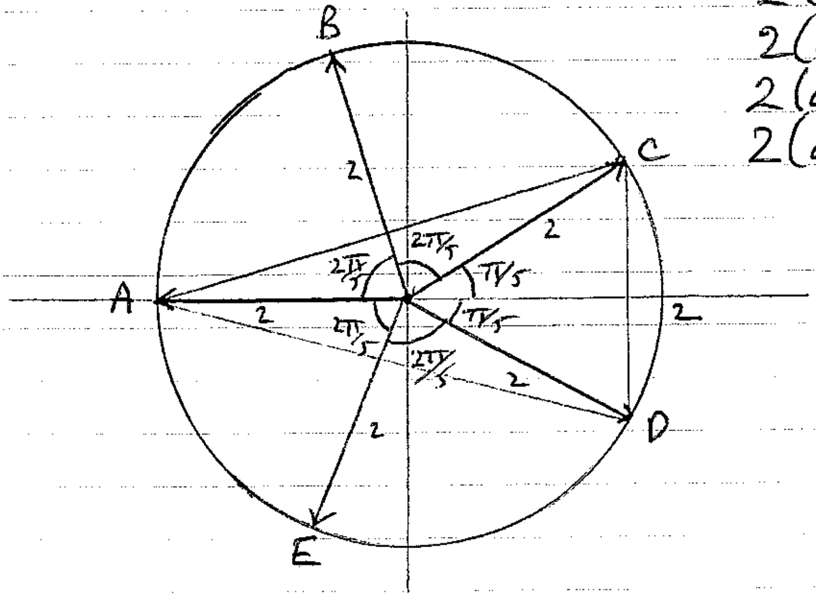
$2\left(\cos\frac{\pi}{5} + i\sin\frac{\pi}{5}\right)$ C

$2\left(\cos\frac{3\pi}{5} + i\sin\frac{3\pi}{5}\right)$ B

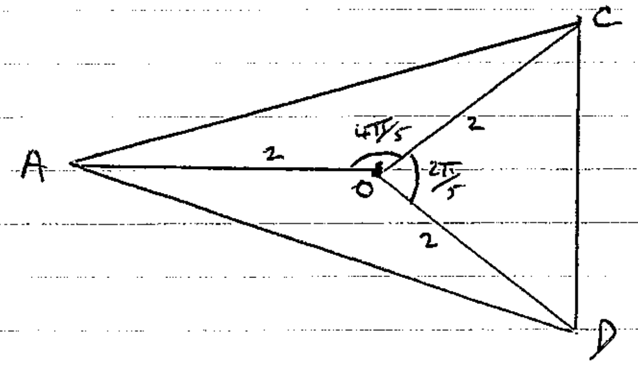
$2\left(\cos\pi + i\sin\pi\right) = -2$ A

$2\left(\cos\left(-\frac{3\pi}{5}\right) + i\sin\left(-\frac{3\pi}{5}\right)\right)$ E

$2\left(\cos\left(-\frac{\pi}{5}\right) + i\sin\left(-\frac{\pi}{5}\right)\right)$ D.



Question 15 (cont)



Total area $= \Delta COD + \Delta AOC + \Delta AOD$

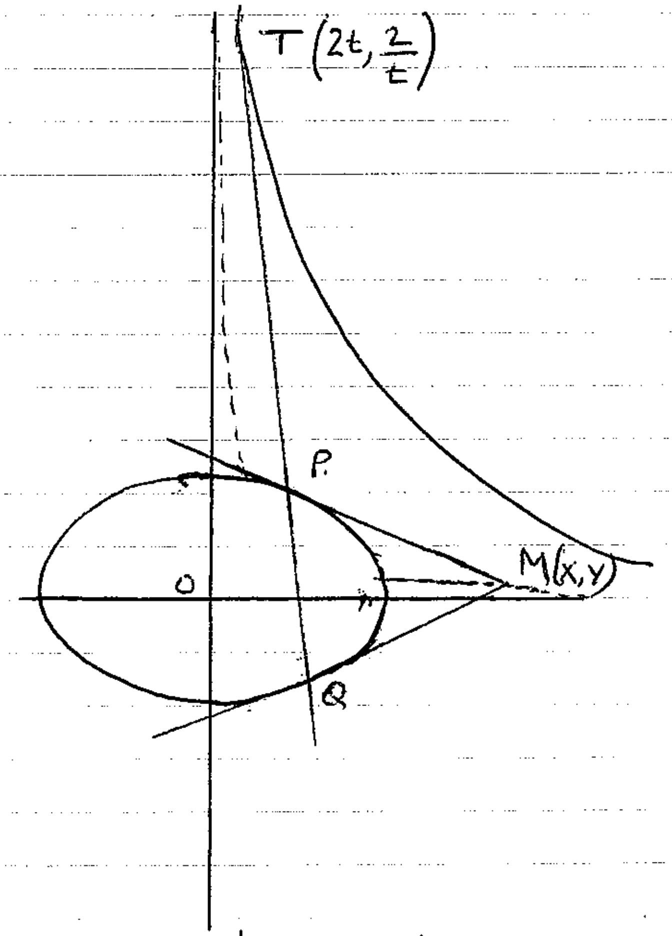
$= \frac{1}{2} \times 2 \times 2 \sin \frac{2\pi}{5} + 2\left(\frac{1}{2} \times 2 \times 2 \sin \frac{4\pi}{5}\right)$

$= 2 \sin \frac{2\pi}{5} + 4 \sin \frac{4\pi}{5}$

$= 1.902113.. + 2.35114$

$= \underline{4.25u^2 \text{ (to 2DP)}}$

c) Let M be the point (X, Y)



The chord of contact from M to the ellipse is given by

$xX + 4yY = 4$ (1)

The tangent from T has equation

$x\left(\frac{2}{t}\right) + 2ty = 8$

or $\frac{x}{t} + ty = 4$ (2)

(1) and (2) are the same line

$\therefore \left.\begin{aligned} X &= 1/t \\ Y &= t/4 \end{aligned}\right\}$ Eliminate $t \Rightarrow XY = \frac{1}{4}$ Locus of M is $\underline{xy = \frac{1}{4}}$

ii) This is a rectangular hyperbola, centred on origin. It will be "tighter" to the axes.
Chords of contact cannot be drawn from inside ellipse.
The curve crosses ellipse when $x^2 + \frac{4}{16x^2} = 4$

$4x^4 - 16x^2 + 1 = 0$

$x^2 = \left(16 \pm \sqrt{240}\right)/8$

$= 2 \pm \frac{\sqrt{15}}{2}$

Restriction on x: $\left.\begin{aligned} x &> \sqrt{2 + \frac{\sqrt{15}}{2}} \\ 0 < x &< \sqrt{2 - \frac{\sqrt{15}}{2}} \end{aligned}\right\}$

And similar on negative side.

Question 16

a) Alan picks first:

Prob. of success $= \underset{\uparrow \text{ 1st pick}}{\frac{1}{n}} + \underset{\uparrow \text{A fails}}{\frac{(n-1)}{n}}\underset{\uparrow \text{B fails}}{\frac{(n-2)}{(n-1)}}\underset{\uparrow \text{A picks}}{\frac{1}{(n-2)}} + \frac{(n-1)}{n}\frac{(n-2)}{(n-1)}\frac{(n-3)}{(n-2)}\frac{(n-4)}{(n-3)}\frac{1}{(n-4)} + \dots$

$\dots + \frac{(n-1)}{n}\frac{(n-2)}{(n-1)} \dots \frac{4}{5} \cdot \frac{3}{4} \cdot \frac{2}{3} \cdot \frac{1}{2} \cdot 1$

Each term is $1/n$, and there are $(n+1)/2$ terms

$\therefore$ Prob $= \underline{\underline{\frac{n+1}{2n}}}$

ii) Prob $= \frac{2}{n} + \frac{\cancel{n-2}}{n}\frac{n-3}{n-1}\frac{2}{\cancel{n-2}} + \dots + \frac{(n-2)}{n}\frac{(n-3)}{(n-1)} \dots \frac{4}{6} \cdot \frac{3}{5} \cdot \frac{2}{4} \cdot \frac{1}{3}\frac{2}{2}$

$= \frac{2}{n}\left(\frac{n-1}{n-1} + \frac{n-3}{n-1} + \frac{n-5}{n-1} + \dots + \frac{1}{n-1}\right)$

$= \frac{2}{n(n-1)}\left(1+3+5+\dots+(n-1)\right)$ ← A.P. with $a=1, d=2$, "n" $= n/2$

$= \frac{2}{n(n-1)}\frac{n}{4}\left(2+\left(\frac{n}{2}-1\right)2\right) = \frac{2}{n(n-1)}\frac{n^2}{4}$

$= \underline{\underline{\frac{n}{2(n-1)}}}$

iii) Prob $= \frac{2}{n} + \frac{\cancel{(n-2)}}{n}\frac{(n-3)}{(n-1)}\frac{2}{\cancel{(n-2)}} + \dots + \frac{(n-2)}{n}\frac{(n-3)}{(n-1)} \dots \frac{5}{7}\frac{4}{6}\frac{3}{5} \cdot \frac{2}{4} \cdot \frac{2}{3}$

$= \frac{2}{n}\left(\frac{n-1}{n-1} + \frac{n-3}{n-1} + \dots + \frac{2}{n-1}\right)$

$= \frac{2}{n(n-1)}\left(2+4+6+\dots+(n-1)\right)$ ← AP $a=2, d=2$ "n" $= \frac{n-1}{2}$

$= \frac{2}{n(n-1)}\frac{(n-1)}{4}\left(4+\left(\left(\frac{n-1}{2}\right)-1\right)2\right)$

$= \frac{2}{n(n-1)}\frac{(n-1)(n+1)}{4}$

$= \underline{\underline{\frac{n+1}{2n}}}$

Question 16 (cont)

b) i) Solve $y = mx \pm \sqrt{a^2m^2+b^2}$ against $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

Sub for y to get quadratic in x.

$$\frac{x^2}{a^2} + \frac{(mx \pm \sqrt{a^2m^2+b^2})^2}{b^2} = 1$$

$$\frac{x^2}{a^2} + \frac{m^2x^2 + a^2m^2+b^2 \pm 2mx\sqrt{m^2a^2+b^2}}{b^2} - 1 = 0$$

$$b^2x^2 + a^2m^2x^2 + a^4m^2 + \cancel{a^2b^2} \pm 2a^2m\sqrt{m^2a^2+b^2}\,x - \cancel{a^2b^2} = 0$$

$$x^2(a^2m^2+b^2) \pm 2a^2m\sqrt{m^2a^2+b^2}\,x + a^4m^2 = 0$$

$$\left(x\sqrt{a^2m^2+b^2} \pm a^2m\right)^2 = 0$$

As this always gives a double root, the lines $y = mx \pm \sqrt{a^2m^2+b^2}$ are always tangents to the ellipse.

ii) Let $P(X, Y)$ be external to the ellipse. Then the gradients of the 2 tangents from P are given by

$$Y = mX \pm \sqrt{a^2m^2+b^2}$$

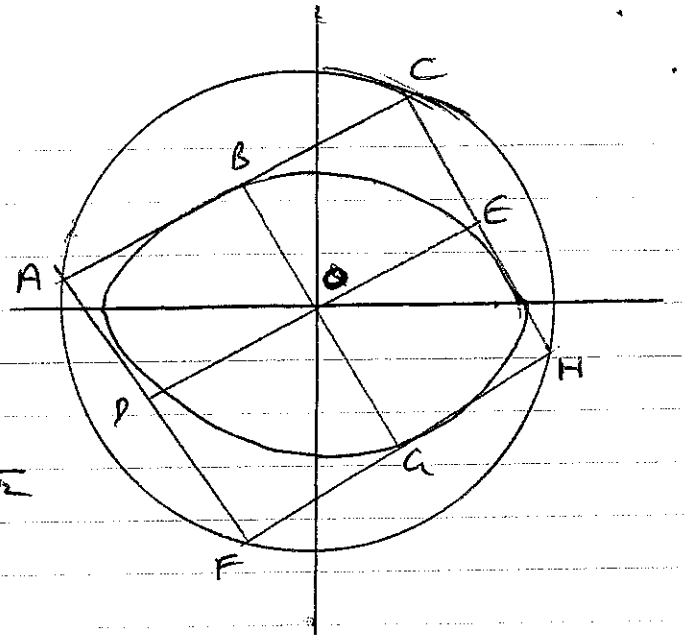
$$Y - mX = \sqrt{a^2m^2+b^2}$$
$$Y^2 + m^2X^2 - 2mXY = a^2m^2+b^2$$
$$m^2(X^2-a^2) - 2mXY + (Y^2-b^2) = 0$$

If these lines are perpendicular, the two roots m_1, m_2 must give $m_1m_2 = -1$. This is the product of the roots.

$$\therefore \frac{Y^2-b^2}{X^2-a^2} = -1$$
$$Y^2 - b^2 = -X^2 + a^2$$
$$\underline{X^2+Y^2 = a^2+b^2}$$

Thus the locus of points where the two tangents are perpendicular is $x^2+y^2 = a^2+b^2$, a circle.

Question 16 (cont)



b) iii)

① AC is line $y = mx + \sqrt{a^2m^2+b^2}$

② FH is line $y = mx - \sqrt{a^2m^2+b^2}$

③ DE is line $y = mx$.

④ CH is line $y = \frac{-x}{m} + \sqrt{\frac{a^2}{m^2}+b^2}$

⑤ AF is line $y = -\frac{x}{m} - \sqrt{\frac{a^2}{m^2}+b^2}$

⑥ BG is line $y = \frac{-x}{m}$

To find length CA, solve ① against the circle.

$$x^2 + \left(mx + \sqrt{a^2m^2+b^2}\right)^2 = a^2+b^2$$

$$x^2(1+m^2) + 2mx\sqrt{a^2m^2+b^2} + a^2m^2 - a^2 = 0$$

If roots are x_1, x_2 then $(x_1-x_2)^2 = (x_1+x_2)^2 - 4x_1x_2$

$$= \frac{4m^2(a^2m^2+b^2)}{(1+m^2)^2} - \frac{4a^2(m^2-1)}{(1+m^2)}$$

$$= \frac{4m^2(a^2m^2+b^2) - 4a^2(m^4-1)}{(1+m^2)^2}$$

$$= \frac{4(m^2b^2+a^2)}{(1+m^2)^2}$$

$(y_1-y_2) = m(x_1-x_2) \quad \therefore (y_1-y_2)^2 = \frac{4m^2(m^2b^2+a^2)}{(1+m^2)^2}$

$\therefore (CA)^2$ is σ (length)^2 $(x_1-x_2)^2 + (y_1-y_2)^2 = \frac{4(m^2b^2+a^2)(1+m^2)}{(1+m^2)^2}$

$$\therefore CA = \sqrt{\frac{4(m^2b^2+a^2)}{(1+m^2)}}$$

Similarly, replacing m with $-\frac{1}{m}$, $CH = \sqrt{\frac{4\left(\frac{b^2}{m^2}+a^2\right)}{\left(1+\frac{1}{m^2}\right)}}$

$$= \sqrt{\frac{4(a^2m^2+b^2)}{(1+m^2)}}$$

$\therefore$ Area is $CA \times CH = \sqrt{\dfrac{4(a^2+m^2b^2)}{1+m^2}}\sqrt{\dfrac{4(a^2m^2+b^2)}{1+m^2}}$

$$= \frac{4\sqrt{(a^2+m^2b^2)(a^2m^2+b^2)}}{1+m^2}$$

iv) $(a-b)^2 \geqslant 0$

$a^2+b^2-2ab \geqslant 0$

$a^2+b^2 \geqslant 2ab$

Substitute $a = \sqrt{m}$, $b = \dfrac{1}{\sqrt{m}}$ (Requires $m > 0$)

$\therefore \; m + \dfrac{1}{m} \geqslant 2\dfrac{\sqrt{m}}{\sqrt{m}}$, i.e. $m + \dfrac{1}{m} \geqslant 2$ for $m > 0$.

v) $$A^2 = 16a^2b^2 + \frac{16(a^2-b^2)^2}{\left(m+\frac{1}{m}\right)^2}$$

Smallest A will be as $m + 1/m \to \infty$, no contribution from second term

$A^2_{min} = 16a^2b^2 \qquad A_{min} = 4ab$

Largest A will be when $m + 1/m = 2$.

$$A^2_{max} = 16a^2b^2 + \frac{16(a^2-b^2)^2}{4} = 16a^2b^2 + 4(a^4+b^4-2a^2b^2)$$

$$= 8a^2b^2 + 4a^4 + 4b^4$$

$$= 4(a^4+b^4+2a^2b^2)$$

$$= 4(a^2+b^2)^2$$

$\therefore A_{max} = 2(a^2+b^2)$