2015 Higher School Certificate Examination Paper
Mathematics Extension 2

Section I

10 marks
Attempt Questions 1-10
Allow about 15 minutes for this section

1 Which conic has eccentricity $\frac{\sqrt{13}}{3}$?

(A) $\frac{x^2}{3}+\frac{y^2}{2}=1$

(B) $\frac{x^2}{3^2}+\frac{y^2}{2^2}=1$

(C) $\frac{x^2}{3}-\frac{y^2}{2}=1$

(D) $\frac{x^2}{3^2}-\frac{y^2}{2^2}=1$

2 What value of z satisfies $z^2=7-24i$?

(A) $4-3i$

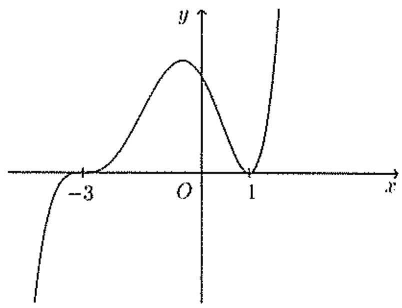
(B) $-4-3i$

(C) $3-4i$

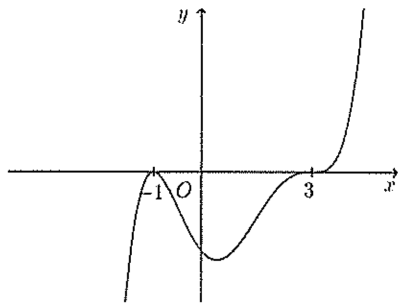
(D) $-3-4i$

3 Which graph best represents the curve $y=(x-1)^2(x+3)^5$?

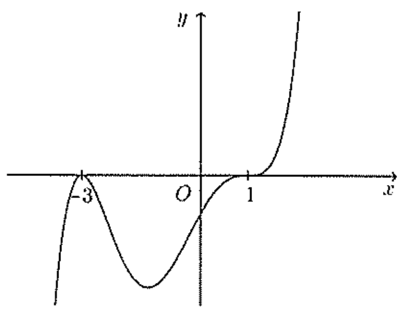
(A)



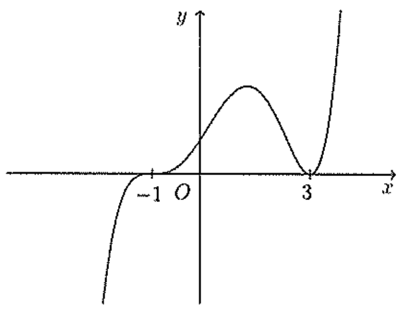
(B)



(C)



(D)



4 The polynomial $x^3 + x^2 - 5x + 3$ has a double root at $x = a$.

What is the value of a?

(A) $-\frac{5}{3}$

(B) -1

(C) 1

(D) $\frac{5}{3}$

5 Given that $z = 1 - i$, which expression is equal to z^3?

(A) $\sqrt{2}\left(\cos\left(\frac{-3\pi}{4}\right) + i\sin\left(\frac{-3\pi}{4}\right)\right)$

(B) $2\sqrt{2}\left(\cos\left(\frac{-3\pi}{4}\right) + i\sin\left(\frac{-3\pi}{4}\right)\right)$

(C) $\sqrt{2}\left(\cos\left(\frac{3\pi}{4}\right) + i\sin\left(\frac{3\pi}{4}\right)\right)$

(D) $2\sqrt{2}\left(\cos\left(\frac{3\pi}{4}\right) + i\sin\left(\frac{3\pi}{4}\right)\right)$

6 Which expression is equal to $\int x^2 \sin x \; dx$?

(A) $-x^2 \cos x - \int 2x \cos x \; dx$

(B) $-2x \cos x + \int x^2 \cos x \; dx$

(C) $-x^2 \cos x + \int 2x \cos x \; dx$

(D) $-2x \cos x - \int x^2 \cos x \; dx$

7 The numbers $1, 2, \ldots n$, for $n \geq 4$, are randomly arranged in a row.

What is the probability that the number 1 is somewhere to the left of the number 2?

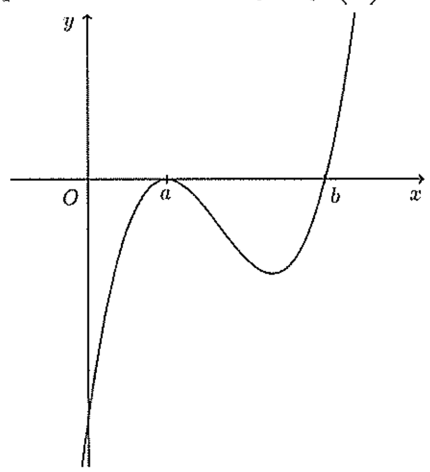
(A) $\frac{1}{2}$

(B) $\frac{1}{n}$

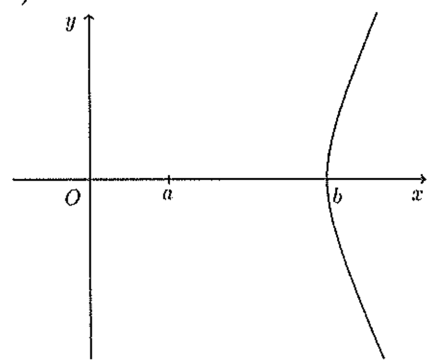
(C) $\frac{1}{2(n-2)!}$

(D) $\frac{1}{2(n-1)!}$

8 The graph of the function $y = f(x)$ is shown.



A second graph is obtained from the function $y = f(x)$.



Which equation best represents the second graph?

(A) $y^2 = |f(x)|$

(B) $y^2 = f(x)$

(C) $y = \sqrt{f(x)}$

(D) $y = f(\sqrt{x})$

9 The complex number z satisfies $|z-1|=1$.

What is the greatest distance that z can be from the point i on the Argand Diagram?

(A) 1

(B) $\sqrt{5}$

(C) $2\sqrt{2}$

(D) $\sqrt{2}+1$

10 Consider the expansion of

$(1+x+x^2+\ldots+x^n)(1+2x+3x^2+\ldots+(n+1)x^n)$

What is the coefficient of x^n when $n=100$?

(A) 4950

(B) 5050

(C) 5151

(D) 5253

Section II

90 marks
Attempt Questions 11-16
Allow about 2 hours 45 minutes for this section

Question 11 (15 marks) **Marks**

(a) Express $\dfrac{4+3i}{2-i}$ in the form $x+iy$, where x and y are real. **2**

(b) Consider the complex numbers

$z=-\sqrt{3}+i$ and $w=3\left(\cos\dfrac{\pi}{7}+i\sin\dfrac{\pi}{7}\right)$.

(i) Evaluate $|z|$. **1**

(ii) Evaluate $\arg(z)$. **1**

(ii) Find the argument of $\dfrac{z}{w}$. **1**

(c) Find A, B and C such that **2**

$$\frac{1}{x(x^2+2)}=\frac{A}{x}+\frac{Bx+C}{x^2+2}.$$

(d) Sketch $\dfrac{x^2}{25}+\dfrac{y^2}{16}=1$ indicating the coordinates of the foci. **2**

(e) Find the value of $\dfrac{dy}{dx}$ at the point $(2,-1)$ on the curve $x+x^2y^3=-2$. **3**

(f) (i) Show that $\cot\theta+\operatorname{cosec}\theta=\cot\left(\dfrac{\theta}{2}\right)$. **2**

(ii) Hence, or otherwise, find **1**

$$\int(\cot\theta+\operatorname{cosec}\theta)\,d\theta.$$

Question 12 (15 marks) **Marks**

(a) The complex number z is such that $|z|=2$ and $\arg(z)=\dfrac{\pi}{4}$.

Plot each of the following complex numbers on the same half-page Argand diagram.

(i) z **1**

(ii) $u=z^2$ **1**

(iii) $v=z^2-\bar{z}$ **1**

(b) The polynomial $P(x)=x^4-4x^3+11x^2-14x+10$ has roots $a+ib$ and $a+2ib$ where a and b are real and $b\neq 0$.

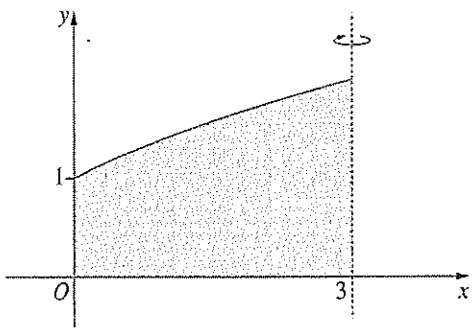
(i) By evaluating a and b, find all the roots of $P(x)$. **3**

(ii) Hence, or otherwise, find one quadratic polynomial with real coefficients that is a factor of $P(x)$. **1**

(c) (i) By writing $\dfrac{(x-2)(x-5)}{x-1}$ in the form $mx+b+\dfrac{a}{x-1}$, find the equation of the oblique asymptote of $y=\dfrac{(x-2)(x-5)}{x-1}$ **2**

(ii) Hence sketch the graph $y=\dfrac{(x-2)(x-5)}{x-1}$, clearly indicating all intercepts and asymptotes. **2**

(d) The diagram shows the graph $y=\sqrt{x+1}$ for $0 \le x \le 3$. The shaded region is rotated about the line $x=3$ to form a solid. **4**



Use the method of cylindrical shells to find the volume of the solid.

Question 13 (15 marks) **Marks**

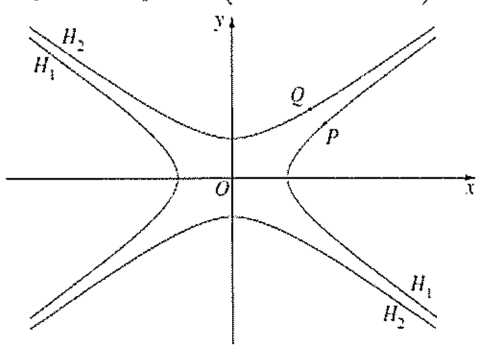
(a) The hyperbolas

$H_1 : \dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$ and $H_2 : \dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = -1$

are shown in the diagram.

Let $P(a\sec\theta, b\tan\theta)$ lie on H_1 as shown on the diagram.

Let Q be the point $(a\tan\theta, b\sec\theta)$.

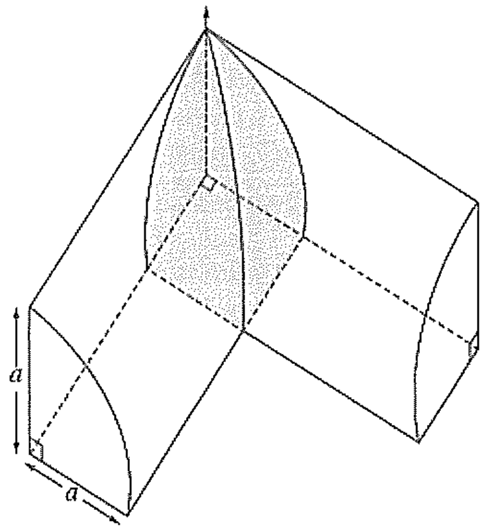


(i) Verify that the coordinates of $Q(a\tan\theta, b\sec\theta)$ satisfy the equation for H_2. **1**

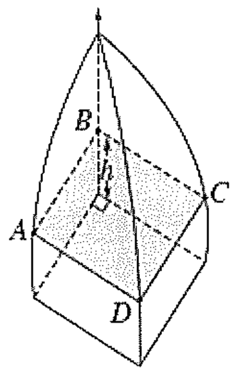
(ii) Show that the equation of the line PQ is $bx+ay=ab(\tan\theta+\sec\theta)$. **2**

(iii) Prove that the area of $\triangle OPQ$ is independent of θ. **3**

(b) Two quarter cylinders, each of radius a, intersect at right angles to form the shaded solid.



A horizontal slice ABCD of the solid is taken at height h from the base. You may assume that ABCD is a square, and is parallel to the base.



(i) Show that $AB=\sqrt{a^2-h^2}$. **1**

(ii) Find the volume of the solid. **2**

(c) A small spherical balloon is released and rises into the air. At time t seconds, it has radius r cm, surface area $S = 4\pi r^2$ and volume $V = \frac{4}{3}\pi r^3$.

As the balloon rises it expands, causing its surface area to increase at a rate of $\left(\frac{4\pi}{3}\right)^{\frac{1}{3}}$ cm^2s^{-1}. As the balloon expands it maintains a spherical shape.

(i) By considering the surface area, show that $\frac{dr}{dt} = \frac{1}{8\pi r}\left(\frac{4}{3}\pi\right)^{\frac{1}{3}}$. **2**

(ii) Show that $\frac{dV}{dt} = \frac{1}{2}V^{\frac{1}{3}}$. **2**

(iii) When the balloon is released its volume is 8000 cm^3. When the volume of the balloon reaches $64\,000 \text{ cm}^3$ it will burst. **2**

How long after it is released will the balloon burst?

Question 14 (15 marks) **Marks**

(a) (i) Differentiate $\sin^{n-1}\theta\cos\theta$, expressing the result in terms of $\sin\theta$ only. **2**

(ii) Hence, or otherwise, deduce that **2**

$$\int_0^{\frac{\pi}{2}} \sin^n\theta \; d\theta = \frac{(n-1)}{n}\int_0^{\frac{\pi}{2}} \sin^{n-2}\theta \; d\theta,$$

for $n > 1$.

(iii) Find $\int_0^{\frac{\pi}{2}} \sin^4\theta \; d\theta$. **1**

(b) The cubic equation $x^3 - px + q = 0$ has roots α, β and γ.

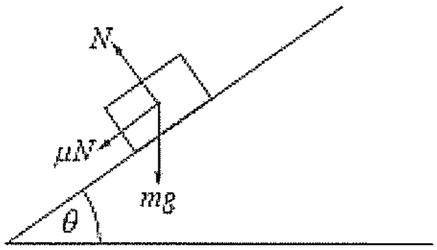
It is given that $\alpha^2 + \beta^2 + \gamma^2 = 16$ and $\alpha^3 + \beta^3 + \gamma^3 = -9$.

(i) Show that $p = 8$. **1**

(ii) Find the value of q. **2**

(iii) Find the value of $\alpha^4 + \beta^4 + \gamma^4$. **2**

(c) A car of mass m is driven at speed v around a circular track of radius r. The track is banked at a constant angle θ to the horizontal, where $0 < \theta < \frac{\pi}{2}$. At the speed v there is a tendency for the car to slide up the track. This is opposed by a frictional force μN, where N is the normal reaction between the car and the track, and $\mu > 0$. The acceleration due to gravity is g.



(i) Show $v^2 = rg\left(\frac{\tan\theta + \mu}{1 - \mu\tan\theta}\right)$. **3**

2

(ii) At the particular speed V, where $V^2 = rg$, there is still a tendency for the car to slide up the track. **2**

Using the result from part (i), or otherwise, show that $\mu < 1$.

Question 15 (15 marks) **Marks**

(a) A particle A of unit mass travels horizontally through a viscous medium. When $t = 0$, the particle is at point O with initial speed u. The resistance on particle A due to the medium is kv^2, where v is the velocity of the particle at time t and k is a positive constant.

When $t = 0$, a second particle B of equal mass is projected vertically upwards from O with the same initial speed u through the same medium. It experiences both a gravitational force and a resistance due to the medium. The resistance on particle B is kw^2, where w is the velocity of the particle B at time t. The acceleration due to gravity is g.

(i) Show that the velocity v of particle A is given by $\frac{1}{v} = kt + \frac{1}{u}$. **2**

(ii) By considering the velocity w of particle B, show that **3**

$$t = \frac{1}{\sqrt{gk}}\left(\tan^{-1}\left(u\sqrt{\frac{k}{g}}\right) - \tan^{-1}\left(w\sqrt{\frac{k}{g}}\right)\right).$$

(iii) Show that the velocity V of particle A when particle B is at rest is given by $\frac{1}{V} = \frac{1}{u} + \sqrt{\frac{k}{g}}\tan^{-1}\left(u\sqrt{\frac{k}{g}}\right)$. **1**

(iv) Hence, if u is very large, explain why $V = \frac{2}{\pi}\sqrt{\frac{g}{k}}$. **1**

(b) Suppose that $x \geq 0$ and n is a positive integer.

(i) Show that $1 - x \leq \frac{1}{1+x} \leq 1$. **2**

(ii) Hence, or otherwise, show that **2**

$$1 - \frac{1}{2n} \leq n\ln\left(1 + \frac{1}{n}\right) \leq 1.$$

(iii) Hence, explain why **1**

$$\lim_{n\to\infty}\left(1 + \frac{1}{n}\right)^n = e.$$

(c) For positive real numbers x and y, $\sqrt{xy} \leq \frac{x+y}{2}$. (Do NOT prove this.)

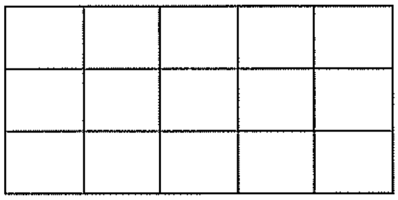
(i) Prove $\sqrt{xy} \leq \sqrt{\frac{x^2 + y^2}{2}}$, for positive real numbers x and y. **1**

(ii) Prove **2**

$$\sqrt[4]{abcd} \leq \sqrt{\frac{a^2 + b^2 + c^2 + d^2}{4}},$$ for positive real numbers a, b, c and d.

Question 16 (15 marks) **Marks**

(a) (i) A table has 3 rows and 5 columns, creating 15 cells as shown. **2**



Counters are to be placed randomly on the table so that there is one counter in each cell. There are 5 identical black counters and 10 identical white counters.

Show that the probability that there is exactly one black counter in each column is $\frac{81}{1001}$.

(ii) The table is extended to have n rows and q columns. There are nq counters, where q are identical black counters and the remainder are identical white counters. The counters are placed randomly on the table with one counter in each cell. **2**

Let P_n be the probability that each column contains exactly one black counter.

Show that $P_n = \frac{n^q}{\binom{nq}{q}}$.

(iii) Find $\lim_{n\to\infty} P_n$. **2**

(b) Let n be a positive integer.

(i) By considering $(\cos\alpha + i\sin\alpha)^{2n}$, show that **2**

$$\cos(2n\alpha) = \cos^{2n}\alpha - \binom{2n}{2}\cos^{2n-2}\alpha\sin^2\alpha + \binom{2n}{4}\cos^{2n-4}\alpha\sin^4\alpha - \ldots + \ldots + (-1)^{n-1}\binom{2n}{2n-2}\cos^2\alpha\sin^{2n-2}\alpha + (-1)^n\sin^{2n}\alpha$$

(ii) Let $T_{2n}(x) = \cos\left(2n\cos^{-1}x\right)$, for $-1 \le x \le 1$. **2**

Show that

$$T_{2n}(x) = x^{2n} - \binom{2n}{2}x^{2n-2}\left(1-x^2\right) + \binom{2n}{4}x^{2n-4}\left(1-x^2\right)^2 + \ldots + (-1)^n\left(1-x^2\right)^n.$$

(iii) By considering the roots of $T_{2n}(x)$, find the value of **3**

$$\cos\left(\frac{\pi}{4n}\right)\cos\left(\frac{3\pi}{4n}\right)\ldots\cos\left(\frac{(4n-1)\pi}{4n}\right)$$

(iv) Prove that **2**

$$1 - \binom{2n}{2} + \binom{2n}{4} - \binom{2n}{6} + \ldots + (-1)^n\binom{2n}{2n} = 2^n\cos\left(\frac{n\pi}{2}\right).$$

End of paper

2015 Higher School Certificate Solutions Mathematics Extension 2

SECTION I

Summary

1 D	3 A	5 B	7 A	9 D
2 A	4 C	6 C	8 B	10 C

SECTION I

1 (D) $e=\dfrac{\sqrt{13}}{3}>1$

$\therefore$ hyperbola thus it is (C) or (D)

$$b^2=a^2\left(e^2-1\right)$$

$$\frac{b^2}{a^2}=e^2-1$$

$$e^2=\frac{b^2}{a^2}+1$$

$$\left(\frac{\sqrt{13}}{3}\right)^2=\frac{b^2}{a^2}+1$$

$$\frac{13}{9}=\frac{b^2}{a^2}+1$$

$$\frac{4}{9}=\frac{b^2}{a^2}$$

$$b=2,\quad a=3$$

$$\frac{x^2}{3^2}-\frac{y^2}{2^2}=1$$

$\therefore$ (D)

2 (A) Let $z=a+ib$

$$z^2=a^2-b^2+2iab$$

But $z^2=7-24i$

$\therefore\ 7=a^2-b^2$

$-24=2ab$

By inspection: $a=\pm 4,\quad b=\mp 3$

$\therefore z=4-3i$.

3 (A) $y=(x-1)^2(x+3)^5$

A double root at $x=1$.

A quintuple root at $x=-3$.

$\therefore$ (A).

4 (C) $f(x)=x^3+x^2-5x+3$

$$f'(x)=3x^2+2x-5$$
$$=(3x+5)(x-1)$$

Try $x=1$:

$$f(1)=(1)^3+(1)^2-5(1)+3=0$$
$$f'(x)=3(1)^2+2(1)-5=0$$

$\therefore$ Double root at $x=1$.

5 (B) $z=1-i$

$$=\sqrt{2}\,cis\left(-\frac{\pi}{4}\right)$$

$$z^3=2\sqrt{2}\,cis\left(-\frac{3\pi}{4}\right).$$

6 (C) Integration by parts, where:

$u=x^2 \qquad v=-\cos x$

$du=2x\,dx \qquad dv=\sin x\,dx$

$$\int x^2\sin x\,dx=2x.(-\cos x)-\int x^2.-\cos x\,dx$$
$$=-2x\cos x+\int x^2\cos x\,dx.$$

7 (A) *Method 1:*

Once the 2 is in place, the 1 can be either to the left or to the right of the 2. Because the 2 can be placed anywhere, the problem is symmetrical.

$\therefore\ P(\text{left of }2)=P(\text{right of }2)=0.5$.

OR

Method 2:
For $n=4$, consider the probability of the 2, then consider the probability of a favourable position for the 1:

2 _ _ _	$\frac{1}{4}\times 0 = 0$
_ 2 _ _	$\frac{1}{4}\times\frac{1}{3}=\frac{1}{12}$
_ _ 2 _	$\frac{1}{4}\times\frac{2}{3}=\frac{1}{6}$
_ _ _ 2	$\frac{1}{4}\times 1=\frac{1}{4}$

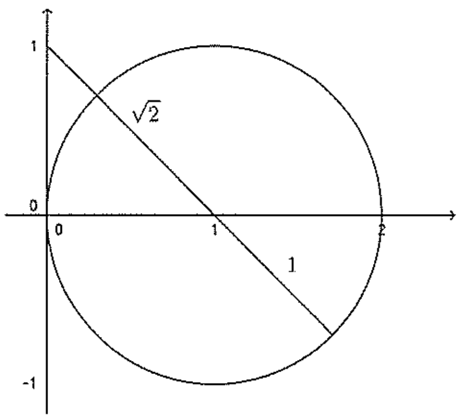
$$\text{Total probability} = \tfrac{1}{4}+\tfrac{1}{6}+\tfrac{1}{12} = \tfrac{1}{2}$$

Similarly, with n numbers, the probability is:

$$\frac{1}{n}\cdot\frac{0}{n-1}+\frac{1}{n}\cdot\frac{1}{n-1}+\frac{1}{n}\cdot\frac{2}{n-1}+\ldots+\frac{1}{n}\cdot\frac{n-1}{n-1}$$
$$=\frac{1}{n}\cdot\frac{1}{n-1}\left(0+1+2+\ldots+(n-1)\right)$$
$$=\frac{1}{n}\cdot\frac{1}{n-1}\times\frac{n}{2}(0+n-1)$$
$$=\frac{1}{n}\cdot\frac{1}{n-1}\cdot\frac{n}{2}(n-1)$$
$$=\frac{1}{2}$$

8 (B) The new function exists for $x=a$ and $x\geq b$.
Thus it is $y=\sqrt{f(x)}$.
But within this domain, it also exists for negative y.
$\therefore\ y=\pm\sqrt{f(x)}$
$y^2=f(x)$.

9 (D)



$$\text{Distance} = \sqrt{1^2+1^2}+1 = \sqrt{2}+1.$$

10 (C) The terms in x^{100} are:
$1\times 101x^{100}+x\times 100x^{99}+x^2\times 99x^{98}+\ldots+x^{100}\times 1$

The coefficients of each term with x^{100} are: $1+2+3+\ldots+101$
$a=1,\quad \ell=101,\quad n=101$

$$S_{101}=\frac{101}{2}(1+101) = 5151.$$

SECTION II

Question 11

(a)

$$\frac{4+3i}{2-i}=\frac{4+3i}{2-i}\times\frac{2+i}{2+i}$$
$$=\frac{8+4i+6i+3(-1)}{4+1}$$
$$=\frac{8-3+i(4+6)}{4+1}$$
$$=\frac{5+10i}{5}$$
$$=1+2i.$$

(b) (i)

$$|z|=\left|-\sqrt{3}+i\right| = \sqrt{\left(\sqrt{3}\right)^2+1^2} = 2.$$

(ii) *Method 1:*

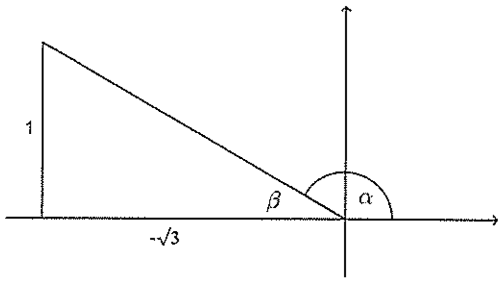
$$\arg(z)=\alpha$$

where $\tan\alpha=\dfrac{1}{-\sqrt{3}}$

and z is in the second quadrant.

$$\arg(z)=\tan^{-1}\left(\frac{1}{-\sqrt{3}}\right) = \pi-\frac{\pi}{6} = \frac{5\pi}{6}.$$

OR



$$\tan\beta = \frac{1}{\sqrt{3}}$$

$$\beta = \frac{\pi}{6}$$

$$\arg(z) = \pi - \frac{\pi}{6} = \frac{5\pi}{6}.$$

(iii) $$\arg\left(\frac{z}{w}\right) = \frac{5\pi}{6} - \frac{\pi}{7} = \frac{29\pi}{42}.$$

(c) $$\frac{1}{x(x^2+2)} = \frac{A}{x} + \frac{Bx+C}{x^2+2}$$

$$1 = A(x^2+2) + (Bx+C)x = (A+B)x^2 + Cx + 2A$$

By equating coefficients:

$$2A = 1$$

$$A = \frac{1}{2}$$

$$C = 0$$

$$A + B = 0$$

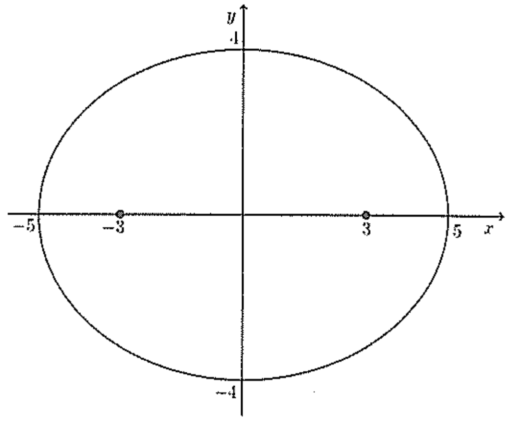
$$B = -A = -\frac{1}{2}$$

$$\therefore A = \frac{1}{2},\quad B = -\frac{1}{2},\quad C = 0.$$

(d) $$b^2 = a^2(1-e^2)$$

$$e^2 = 1 - \frac{16}{25} = \frac{9}{25}, \therefore\ e = \frac{3}{5}.$$

The foci $(\pm ae, 0)$ are $(\pm 3, 0)$:



(e) $$x + x^2y^3 = -2$$

$$1 + 2xy^3 + x^2.3y^2.\frac{dy}{dx} = 0$$

$$3x^2y^2\frac{dy}{dx} = -1 - 2xy^3$$

$$\frac{dy}{dx} = \frac{-1-2xy^3}{3x^2y^2}$$

At $(2,1)$:

$$\frac{dy}{dx} = \frac{-1-2xy^3}{3x^2y^2} = \frac{-1-2(2)(-1)^3}{3(2)^2(-1)^2} = \frac{-1+4}{12} = \frac{1}{4}.$$

(f) (i) *Method 1*

$$\cot\theta + \operatorname{cosec}\theta = \frac{\cos\theta}{\sin\theta} + \frac{1}{\sin\theta} = \frac{\cos\theta + 1}{\sin\theta} = \frac{2\cos^2\frac{\theta}{2}}{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}} = \frac{\cos\frac{\theta}{2}}{\sin\frac{\theta}{2}} = \cot\frac{\theta}{2}.$$

OR

Method 2

Let $t = \tan\dfrac{\theta}{2}$:

$$\begin{aligned}\cot\theta + \operatorname{cosec}\theta &= \frac{1}{\tan\theta} + \frac{1}{\sin\theta} \\ &= \frac{1-t^2}{2t} + \frac{1+t^2}{2t} \\ &= \frac{2}{2t} \\ &= \frac{1}{t} \\ &= \cot\frac{\theta}{2}.\end{aligned}$$

(ii)
$$\begin{aligned}\int(\cot\theta + \operatorname{cosec}\theta)\,d\theta &= \int \cot\frac{\theta}{2}\,d\theta \\ &= \int \frac{\cos\frac{\theta}{2}}{\sin\frac{\theta}{2}}\,d\theta \\ &= 2\int \frac{\frac{1}{2}\cos\frac{\theta}{2}}{\sin\frac{\theta}{2}}\,d\theta \\ &= 2\ln\left(\sin\frac{\theta}{2}\right) + C.\end{aligned}$$

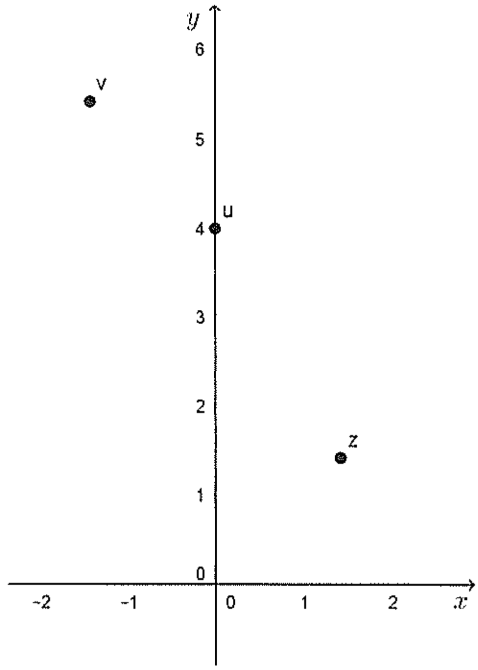
Question 12

(a)

$z = 2\operatorname{cis}\dfrac{\pi}{4} = \sqrt{2} + \sqrt{2}i$

$u = z^2 = 4\operatorname{cis}\dfrac{\pi}{2} = 4i$

$$\begin{aligned}v &= z^2 - \overline{z} \\ &= 4i - \left(\sqrt{2} - \sqrt{2}i\right) \\ &= -\sqrt{2} + \left(4 + \sqrt{2}\right)i\end{aligned}$$



(b) (i) The roots will be in conjugate pairs.

$\therefore$ The roots are $a \pm ib$ and $a \pm 2ib$.

$$\begin{aligned}\Sigma\alpha &= (a+ib)+(a-ib)+(a+2ib)+(a-2ib) \\ 4 &= 4a \\ a &= 1\end{aligned}$$

$$\begin{aligned}\alpha\beta\gamma\delta &= (a+ib)(a-ib)(a+2ib)(a-2ib) \\ 10 &= \left(a^2+b^2\right)\left(a^2+4b^2\right) \\ &= \left(1^2+b^2\right)\left(1^2+4b^2\right) \\ &= 1+b^2+4b^2+4b^4 \\ &= 1+5b^2+4b^4 \\ 0 &= 4b^4+5b^2-9 \\ &= \left(b^2-1\right)\left(4b^2+9\right) \\ b &= \pm 1,\quad \pm\frac{3}{2}i\end{aligned}$$

But b is real.

$\therefore a = 1, \quad b = \pm 1$

$\therefore$ The roots are $1 \pm i$ and $1 \pm 2i$.

(ii) *Method 1:*

$$\begin{aligned}P(x) &= \left(x-(1+i)\right)\left(x-(1-i)\right)\times \\ &\qquad \left(x-(1+2i)\right)\left(x-(1-2i)\right) \\ &= \left(x^2-2x+2\right)\left(x^2-2x+5\right)\end{aligned}$$

$\therefore\ x^2-2x+2$ or x^2-2x+5 is a quadratic factor of $P(x)$.

OR

Method 2:
Let the roots be $\alpha = 1+i$ and $\beta = 1-i$.

$$\alpha+\beta = 2$$
$$\alpha\beta = 2$$

$\therefore\ x^2-2x+2$ is a quadratic factor of $P(x)$.

OR

Method 3:
Let the roots be $\alpha = 1+2i$ and $\beta = 1-2i$.

$$\alpha+\beta = 2$$
$$\alpha\beta = 5$$

$\therefore\ x^2-2x+5$ is a quadratic factor of $P(x)$.

(c) (i) $\dfrac{(x-2)(x-5)}{x-1} = \dfrac{x^2-7x+10}{x-1}$

$$\begin{array}{r|l} & x-6 \\ x-1 & x^2-7x+10 \\ & \underline{x^2-x} \\ & -6x+10 \\ & \underline{-6x+6} \\ & 4 \end{array}$$

$$\frac{(x-2)(x-5)}{x-1} = \frac{x^2-7x+10}{x-1} = x-6+\frac{4}{x-1}$$

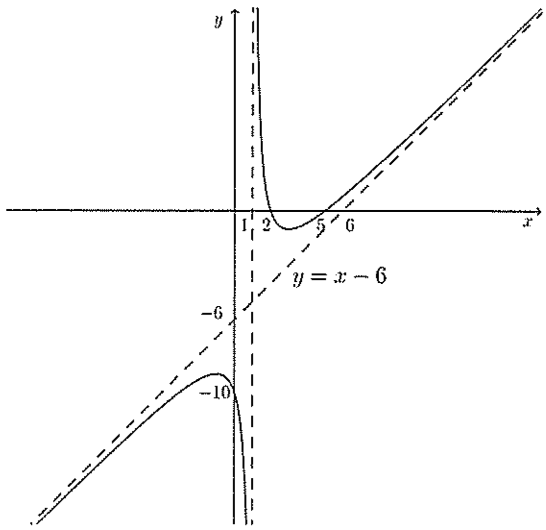
Equation of the oblique asymptote is $y = x-6$.

(ii) Vertical asymptote: $x=1$

Oblique asymptote: $y=x-6$

x-intercepts: $x=2,\quad 5$

y-intercept: $y=-10$



(d) $\delta V = 2\pi r h \delta x$

where $r = 3-x$, $h = y$, $y=\sqrt{x+1}$

$$\delta V = 2\pi(3-x)\,y\,\delta x = 2\pi(3-x)\sqrt{x+1}\,\delta x$$

$$V = 2\pi\int_0^3 (3-x)\sqrt{x+1}\ dx$$

$u^2 = x+1 \quad \Rightarrow x = u^2-1$

$3-x = 4-u^2$

Let $2u\,du = dx$

when $x=3,\quad u=4$

when $x=0,\quad u=1$

$$V = 2\pi\int_0^3 (3-x)\sqrt{x+1}\ dx$$
$$= 2\pi\int_1^2 \left(4-u^2\right)u\ 2u\ du$$
$$= 4\pi\int_1^2 \left(4-u^2\right)u^2\ du$$
$$= 4\pi\int_1^2 \left(4u^2-u^4\right)du$$
$$= 4\pi\left[\frac{4u^3}{3}-\frac{u^5}{5}\right]_1^2$$
$$= 4\pi\left(\frac{32}{3}-\frac{32}{5}-\left(\frac{4}{3}-\frac{1}{5}\right)\right)$$

$$V = 4\pi\left(\frac{28}{3} - \frac{31}{5}\right)$$
$$= 4\pi\left(\frac{140}{15} - \frac{93}{15}\right)$$
$$= 4\pi \times \left(\frac{47}{15}\right)$$
$$= \frac{188\pi}{15} \text{ units}^3.$$

Question 13

(a) (i) $LHS = \frac{x^2}{a^2} - \frac{y^2}{b^2}$

$$= \frac{(a\tan\theta)^2}{a^2} - \frac{(b\sec\theta)^2}{b^2}$$
$$= \tan^2\theta - \sec^2\theta$$
$$= \tan^2\theta - (1 + \tan^2\theta)$$
$$= -1$$
$$= RHS$$

$\therefore Q$ lies on H_2.

(ii) $m = \frac{b\sec\theta - b\tan\theta}{a\tan\theta - a\sec\theta}$

$$= \frac{b(\sec\theta - \tan\theta)}{a(\tan\theta - \sec\theta)}$$
$$= \frac{-b(\tan\theta - \sec\theta)}{a(\tan\theta - \sec\theta)}$$
$$= -\frac{b}{a}$$

$$y - y_1 = m(x - x_1)$$
$$y - b\sec\theta = -\frac{b}{a}(x - a\tan\theta)$$
$$ay - ab\sec\theta = -bx + ab\tan\theta$$
$$bx + ay = ab\tan\theta + ab\sec\theta$$
$$bx + ay = ab(\tan\theta + \sec\theta).$$

(iii) Use PQ as the base of the triangle:

$$PQ = \sqrt{(a\sec\theta - a\tan\theta)^2 + (b\tan\theta - b\sec\theta)^2}$$
$$= \sqrt{a^2(\sec\theta - \tan\theta)^2 + (-b)^2(\sec\theta - \tan\theta)^2}$$
$$= |\sec\theta - \tan\theta|\sqrt{a^2 + b^2}$$

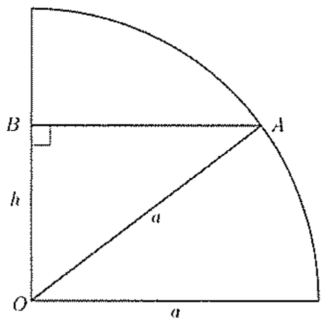
Let h be the perpendicular distance from O to PQ.

$$h = \left|\frac{b(0) + a(0) - ab(\tan\theta + \sec\theta)}{\sqrt{a^2 + b^2}}\right|$$
$$= \left|\frac{ab(\tan\theta + \sec\theta)}{\sqrt{a^2 + b^2}}\right|$$

$$A_{\Delta OPQ} = \frac{1}{2} \times |\sec\theta - \tan\theta|\sqrt{a^2 + b^2} \times \left|\frac{ab(\tan\theta + \sec\theta)}{\sqrt{a^2 + b^2}}\right|$$
$$= \frac{1}{2} \times ab\left|-\tan^2\theta + \sec^2\theta\right|$$
$$= \frac{1}{2} \times ab \times 1$$
$$= \frac{ab}{2}$$

This area is independent of θ.

(b) (i) The view from the side where A and B are situated is:



$$OA^2 = AB^2 + OB^2$$
$$AB^2 = OA^2 - OB^2$$
$$= a^2 - h^2$$
$$AB = \sqrt{a^2 - h^2}.$$

(ii) $V=\int_0^a\left(\sqrt{a^2-h^2}\right)^2\ dh$

$=\int_0^a\left(a^2-h^2\right)\ dh$

$=\left[a^2h-\frac{h^3}{3}\right]_0^a$

$=a^3-\frac{a^3}{3}$

$=\frac{2a^3}{3}$ units^3.

(c) (i) $\frac{dS}{dt}=\left(\frac{4\pi}{3}\right)^{\frac{1}{3}}$ (given)

$S=4\pi r^2$

$\frac{dS}{dr}=8\pi r$

$\frac{dS}{dt}=\frac{dS}{dr}\times\frac{dr}{dt}$

$\left(\frac{4\pi}{3}\right)^{\frac{1}{3}}=8\pi r\times\frac{dr}{dt}$

$\frac{dr}{dt}=\frac{1}{8\pi r}\left(\frac{4\pi}{3}\right)^{\frac{1}{3}}.$

(ii) $V=\frac{4}{3}\pi r^3$

$\frac{dV}{dr}=4\pi r^2$

$\frac{dV}{dt}=\frac{dV}{dr}\times\frac{dr}{dt}$

$=4\pi r^2\times\frac{1}{8\pi r}\left(\frac{4\pi}{3}\right)^{\frac{1}{3}}$

$=\frac{r}{2}\left(\frac{4\pi}{3}\right)^{\frac{1}{3}}$

$=\frac{1}{2}\left(\frac{4\pi r^3}{3}\right)^{\frac{1}{3}}$

$=\frac{1}{2}V^{\frac{1}{3}}.$

(iii) $\frac{dV}{dt}=\frac{1}{2}V^{\frac{1}{3}}$

$2\,dV=V^{\frac{1}{3}}\,dt$

$2V^{-\frac{1}{3}}\,dV=1\,dt$

$\int 2V^{-\frac{1}{3}}\ dV=\int\ dt$

$\int\ dt=\int 2V^{-\frac{1}{3}}\ dV$

$t=3V^{\frac{2}{3}}+C$

When $t=0,\quad V=8000$:

$t=3V^{\frac{2}{3}}+C$

$0=3(8000)^{\frac{2}{3}}+C$

$0=3\times400+C$

$C=-1200$

$t=3V^{\frac{2}{3}}-1200$

When $V=64\,000$:

$t=3V^{\frac{2}{3}}-1200$

$=3(64\,000)^{\frac{2}{3}}-1200$

$=3(40)^2-1200$

$=3600$ seconds

$=1$ hour.

Question 14

(a) (i) $P(\theta)=\sin^{n-1}\theta\cos\theta$

$P'(\theta)=(n-1)\sin^{n-2}\theta.\cos\theta.\cos\theta$

$-\sin^{n-1}\theta\sin\theta$

$=(n-1)\sin^{n-2}\theta\cos^2\theta-\sin^n\theta$

$=(n-1)\sin^{n-2}\theta\left(1-\sin^2\theta\right)-\sin^n\theta$

$=(n-1)\sin^{n-2}\theta-(n-1)\sin^n\theta-\sin^n\theta$

$=(n-1)\sin^{n-2}\theta-\sin^n\theta\left((n-1)+1\right)$

$=(n-1)\sin^{n-2}\theta-n\sin^n\theta$

(iv) When $u \to \infty$, $\dfrac{1}{u} \to 0$

and $\tan^{-1} u\sqrt{\dfrac{k}{g}} \to \dfrac{\pi}{2}$

$$\therefore \frac{1}{V} \to \sqrt{\frac{k}{g}} \times \frac{\pi}{2}$$

$$V = \sqrt{\frac{g}{k}} \times \frac{2}{\pi}$$

$$= \frac{2}{\pi}\sqrt{\frac{g}{k}}.$$

(b) (i) For $x \geq 0$:

$$1 - x^2 \leq 1$$

$$(1-x)(1+x) \leq 1$$

$$1 - x \leq \frac{1}{1+x} \quad \text{since } 1+x > 0$$

Also $\dfrac{1}{1+x} \leq 1$

since the denominator > numerator.

Combining the two parts:

$$1 - x \leq \frac{1}{1+x} \leq 1.$$

(ii) $$\int_0^{\frac{1}{n}} 1 - x \, dx \leq \int_0^{\frac{1}{n}} \frac{1}{1+x} \, dx \leq \int_0^{\frac{1}{n}} dx$$

$$\left[x - \frac{x^2}{2}\right]_0^{\frac{1}{n}} \leq \left[\ln(1+x)\right]_0^{\frac{1}{n}} \leq [x]_0^{\frac{1}{n}}$$

$$\frac{1}{n} - \frac{1}{2n^2} \leq \ln\left(1 + \frac{1}{n}\right) \leq \frac{1}{n}$$

Multiply by n (n>0):

$$1 - \frac{1}{2n} \leq n\ln\left(1 + \frac{1}{n}\right) \leq 1.$$

(iii) $n \to \infty$, $\dfrac{1}{2n} \to 0$

From part (ii):

$$1 \leq n\ln\left(1 + \frac{1}{n}\right) \leq 1$$

$$\therefore \; n\ln\left(1 + \frac{1}{n}\right) \to 1$$

$$n\ln\left(1 + \frac{1}{n}\right) = 1$$

$$\ln\left(1 + \frac{1}{n}\right)^n = \ln e$$

$$\left(1 + \frac{1}{n}\right)^n = e.$$

(c) (i) *Method 1:*

$$(x-y)^2 \geq 0$$

$$x^2 - 2xy + y^2 \geq 0$$

$$x^2 + y^2 \geq 2xy$$

$$xy \leq \frac{x^2 + y^2}{2} \quad ①$$

$$\sqrt{xy} \leq \sqrt{\frac{x^2 + y^2}{2}}.$$

OR

Method 2:

Given $x > 0$ and $y > 0$:

$$x + y = \sqrt{(x+y)^2}$$

$$= \sqrt{x^2 + y^2 + 2xy}$$

$$\leq \sqrt{x^2 + y^2}$$

$$\therefore \frac{x+y}{2} \leq \frac{\sqrt{x^2 + y^2}}{2}$$

$$\sqrt{xy} \leq \frac{x+y}{2} \quad \text{(given)}$$

$$\leq \frac{\sqrt{x^2 + y^2}}{2} \quad \text{(above)}$$

$$\leq \sqrt{\frac{x^2 + y^2}{2}}$$

$$\therefore \sqrt{xy} \leq \sqrt{\frac{x^2 + y^2}{2}}.$$

(ii) *Method 1:*

From ① above:

$$ab \leq \frac{a^2 + b^2}{2}$$

and $cd \leq \dfrac{c^2 + d^2}{2}$

$$\therefore\ ab+cd \le \frac{a^2+b^2}{2}+\frac{c^2+d^2}{2}$$

$$\le \frac{a^2+b^2+c^2+d^2}{2} \quad ②$$

$$\sqrt{abcd}=\sqrt{(ab)+(cd)}$$

$$\le \frac{ab+cd}{2} \quad \text{(by given inequality)}$$

$$\le \frac{a^2+d^2+c^2+d^2}{4} \quad \text{from ②}$$

Taking square roots:

$$\sqrt[4]{abcd} \le \sqrt{\frac{a^2+b^2+c^2+d^2}{4}}.$$

OR

Method 2:
From part (i):

$$\sqrt{ab} \le \sqrt{\frac{a^2+b^2}{2}}$$

$$\sqrt{cd} \le \sqrt{\frac{c^2+d^2}{2}}$$

$$\sqrt{\sqrt{ab}\sqrt{cd}} \le \sqrt{\frac{\left(\sqrt{\frac{a^2+b^2}{2}}\right)^2+\left(\sqrt{\frac{c^2+d^2}{2}}\right)^2}{2}}$$

$$\le \sqrt{\frac{\frac{a^2+b^2}{2}+\frac{c^2+d^2}{2}}{2}}$$

$$\sqrt[4]{abcd} \le \sqrt{\frac{a^2+b^2+c^2+d^2}{4}}.$$

Question 16

(a) (i) There is one black counter in each column so there are 3 ways of placing a black counter in a column, and then there are 5 columns giving 3^5 ways.

The total number of ways of placing 5 black counters is:

$$^{15}C_5 \times {}^{10}C_{10} = {}^{15}C_5$$

$$P(\text{black in each column}) = \frac{3^5}{^{15}C_5}$$

$$= \frac{243}{3003}$$

$$= \frac{81}{1001}.$$

(ii) Using similar logic to part (i):
There are n ways of placing a black counter in a column, and then there are q columns giving n^q ways.

The total number of ways of placing q black counters and nq-q white counters in nq cells is:

$$^{nq}C_q \times {}^{nq-q}C_{nq-q} = {}^{nq}C_q$$

$$P(\text{black in each column}) = \frac{n^q}{^{nq}C_q}$$

$$= \frac{n^q}{\binom{nq}{q}}.$$

(iii) $$P_n = \frac{n^q}{\binom{nq}{q}}$$

$$= \frac{n^q}{\frac{(nq)!}{(nq-q)!q!}}$$

$$= \frac{n^q(nq-q)!q!}{(nq)!}$$

$$= \frac{n^q q!(nq-q)!}{(nq)(nq-1)\ldots(nq-q+1)(nq-q)!}$$

$$= \frac{n^q q!}{(nq)(nq-1)\ldots(nq-q+1)}$$

$$= \frac{n^q q!}{n^q\left(q\left(q-\frac{1}{n}\right)\left(q-\frac{2}{n}\right)\ldots\left(q-\frac{q-1}{n}\right)\right)}$$

$$= \frac{q!}{q\left(q-\frac{1}{n}\right)\left(q-\frac{2}{n}\right)\ldots\left(q-\frac{q-1}{n}\right)}$$

$$\lim_{n\to\infty} P_n = \frac{q!}{q^q}$$

(b) (i) By De Moivre's theorem:

$$(\cos\alpha + i\sin\alpha)^{2n} = \cos 2n\alpha + i\sin 2n\alpha$$

By expanding:

$$\begin{aligned}(\cos\alpha + i\sin\alpha)^{2n} = \cos^{2n}\alpha &+ \binom{2n}{1}\cos^{2n-1}\alpha(i\sin\alpha) \\ &+ \binom{2n}{2}\cos^{2n-2}\alpha(i\sin\alpha)^2 \\ &+ \binom{2n}{3}\cos^{2n-3}\alpha(i\sin\alpha)^3 \\ &\dots \\ &+ \binom{2n}{2n-1}\cos\alpha(i\sin\alpha)^{2n-1} \\ &+ (i\sin\alpha)^{2n}\end{aligned}$$

Consider the real parts (allowing for even powers of i, $i^2=-1$, $i^4=1$):

$$\begin{aligned}\cos 2n\alpha = \cos^{2n}\alpha &- \binom{2n}{2}\cos^{2n-2}\alpha\sin^2\alpha \\ &+ \binom{2n}{4}\cos^{2n-4}\alpha\sin^4\alpha \\ &\dots \\ &(-1)^{n-1}\binom{2n}{2n-2}\cos^2\alpha\sin^{2n-2}\alpha \\ &+ (-1)^n\sin^{2n}\alpha.\end{aligned}$$

(ii)

$$T_{2n}(x) = \cos(2n\cos^{-1}x)$$

$$\therefore\ \alpha = \cos^{-1}x$$

and $x = \cos\alpha$

Also $\sin^2\alpha = 1-\cos^2\alpha$

$$= 1 - x^2$$

$$\begin{aligned}\cos 2n(\cos^{-1}x) = \cos^{2n}(\cos^{-1}x) &- \binom{2n}{2}\cos^{2n-2}(\cos^{-1}x)\sin^2\alpha \\ &+ \binom{2n}{4}\cos^{2n-4}(\cos^{-1}x)\sin^4\alpha \\ &\dots \\ &+ (-1)^n\sin^{2n}\alpha.\end{aligned}$$

This then becomes:

$$\begin{aligned}T_{2n}(x) = x^{2n} &- \binom{2n}{2}x^{2n-2}(1-x^2) \\ &+ \binom{2n}{4}x^{2n-4}(1-x^2)^2 \\ &\dots \\ &+ (-1)^n(1-x^2)^n.\end{aligned}$$

(iii) Let $\cos(2n\cos^{-1}x) = 0$

$$2n\cos^{-1}x = \frac{\pi}{2}, \frac{3\pi}{2}, \dots, \frac{(2k-1)\pi}{2}, \quad k = 1,2,3,\dots 2n$$

$$\cos^{-1}x = \frac{\pi}{4n}, \frac{3\pi}{4n}, \dots, \frac{(2k-1)\pi}{4n}, \quad k = 1,2,3,\dots 2n$$

$$x = \cos\left(\frac{\pi}{4n}\right), \cos\left(\frac{3\pi}{4n}\right), \dots, \cos\left(\frac{(4n-1)\pi}{4n}\right)$$

These are the roots of $T_{2n}(x)$.

The product of the roots:

$$\cos\left(\frac{\pi}{4n}\right)\cos\left(\frac{3\pi}{4n}\right)\dots\cos\left(\frac{(4n-1)\pi}{4n}\right) \quad ①$$

From part (ii), setting $T_{2n}(x) = 0$:

$$0 = x^{2n} - \binom{2n}{2}x^{2n-2}(1-x^2) + \binom{2n}{4}x^{2n-4}(1-x^2)^2$$

$$\dots + (-1)^n(1-x^2)^n$$

The constant term will be:

$$\frac{(-1)^{2n}(-1)^n}{\text{coefficient of } x^{2n}} = \frac{(-1)^n}{1 + \binom{2n}{2} + \binom{2n}{4} + \dots + \binom{2n}{2n}} \quad ②$$

To find the value of the denominator:

$$(1+x)^{2n} = 1 + \binom{2n}{1}x + \binom{2n}{2}x^2 + \dots + \binom{2n}{2n}x^{2n}$$

For $x = 1$:

$$(1+1)^{2n} = 1 + \binom{2n}{1}1 + \binom{2n}{2}1^2 + \dots + \binom{2n}{2n}1^{2n}$$

$$2^{2n} = 1 + \binom{2n}{1} + \binom{2n}{2} + \dots + \binom{2n}{2n}$$

For $x = -1$:

$$(1-1)^{2n} = 1 + \binom{2n}{1}(-1) + \binom{2n}{2}(-1)^2 + \dots + \binom{2n}{2n}(-1)^{2n}$$

$$0 = 1 - \binom{2n}{1} + \binom{2n}{2} + \dots + (-1)^n\binom{2n}{2n}$$

Adding the 2 previous results:

$$2^{2n}=2\left(1+\binom{2n}{2}+\binom{2n}{4}+\ldots+\binom{2n}{2n}\right)$$

Divide by 2:

$$2^{2n-1}=1+\binom{2n}{2}+\binom{2n}{4}+\ldots+\binom{2n}{2n} \quad ③$$

Combining ①, ②, ③:

$$\cos\left(\frac{\pi}{4n}\right)\cos\left(\frac{3\pi}{4n}\right)\ldots\cos\left(\frac{(4n-1)\pi}{4n}\right)$$

$$=\frac{(-1)^n}{1+\binom{2n}{2}+\binom{2n}{4}+\ldots+\binom{2n}{2n}}$$

$$=\frac{(-1)^n}{2^{2n-1}}.$$

(iv) From part (ii):

$$\cos 2n\left(\cos^{-1}x\right)=x^{2n}-\binom{2n}{2}x^{2n-2}\left(1-x^2\right)$$
$$+\binom{2n}{4}x^{2n-4}\left(1-x^2\right)^2$$
$$\ldots$$
$$+(-1)^n\left(1-x^2\right)^n$$

For $x=\frac{1}{\sqrt{2}}$ and using part (ii):

$$\cos 2n\left(\cos^{-1}\frac{1}{\sqrt{2}}\right)=\left(\frac{1}{\sqrt{2}}\right)^{2n}$$
$$-\binom{2n}{2}\left(\frac{1}{\sqrt{2}}\right)^{2n-2}\left(1-\left(\frac{1}{\sqrt{2}}\right)^2\right)$$
$$+\binom{2n}{4}\left(\frac{1}{\sqrt{2}}\right)^{2n-4}\left(1-\left(\frac{1}{\sqrt{2}}\right)^2\right)^2$$
$$\ldots$$
$$+(-1)^n\left(1-\left(\frac{1}{\sqrt{2}}\right)^2\right)^n$$

$$\cos\left(2n\frac{\pi}{4}\right)=\left(\frac{1}{2}\right)^n$$
$$-\binom{2n}{2}\left(\frac{1}{2}\right)^{n-1}\left(1-\left(\frac{1}{2}\right)\right)$$
$$+\binom{2n}{4}\left(\frac{1}{2}\right)^{n-2}\left(1-\left(\frac{1}{2}\right)\right)^2$$
$$\ldots$$
$$+(-1)^n\left(1-\left(\frac{1}{2}\right)\right)^n$$

$$\cos\left(\frac{n\pi}{2}\right)=\left(\frac{1}{2}\right)^n-\binom{2n}{2}\left(\frac{1}{2}\right)^n+\binom{2n}{4}\left(\frac{1}{2}\right)^n$$
$$\ldots+(-1)^n\binom{2n}{2n}\left(\frac{1}{2}\right)^n$$
$$=\frac{1}{2^n}\left[1-\binom{2n}{2}+\binom{2n}{4}+\ldots+(-1)^n\binom{2n}{2n}\right]$$

Multiply by 2^n:

$$\therefore 1-\binom{2n}{2}+\binom{2n}{4}+\ldots+(-1)^n\binom{2n}{2n}=2^n\cos\left(\frac{n\pi}{2}\right).$$

End of Mathematics Extension 2 solutions