Name: ______________________

Teacher: ______________________

Class: ______________________

FORT STREET HIGH SCHOOL

2014

HIGHER SCHOOL CERTIFICATE COURSE

ASSESSMENT TASK 3: TRIAL HSC

Mathematics Extension 2

Time allowed: 3 hours
(plus 5 minutes reading time)

Syllabus Outcomes	Assessment Area Description and Marking Guidelines	Questions
	Chooses and applies appropriate mathematical techniques in order to solve problems effectively	1-11
E2, E3	Applies appropriate strategies to construct arguments and proofs in the areas of complex numbers and polynomials	12
E4, E6	Uses efficient techniques for the algebraic manipulation of conic sections and determining features of a wide variety of graphs	13
E7, E8	Applies further techniques of integration, such as slicing and cylindrical shells, integration by parts and recurrence formulae, to problems	14
E5	Uses ideas and techniques of calculus to solve problems in mechanics involving resolution of forces, resisted motion and circular motion	15
E2-E8	Synthesises mathematical processes to solve harder problems and communicates solutions in an appropriate form	16

Total Marks 100

Section I 10 marks
Multiple Choice, attempt all questions.
Allow about 15 minutes for this section.
Section II 90 Marks
Attempt Questions 11-16.
Allow about 2 hours 45 minutes for this section.

General Instructions:

- Questions 11-16 are to be started in a new booklet.
- The marks allocated for each question are indicated.
- In Questions 11 – 16, show relevant mathematical reasoning and/or calculations.
- Marks may be deducted for careless or badly arranged work.
- Board – approved calculators may be used.

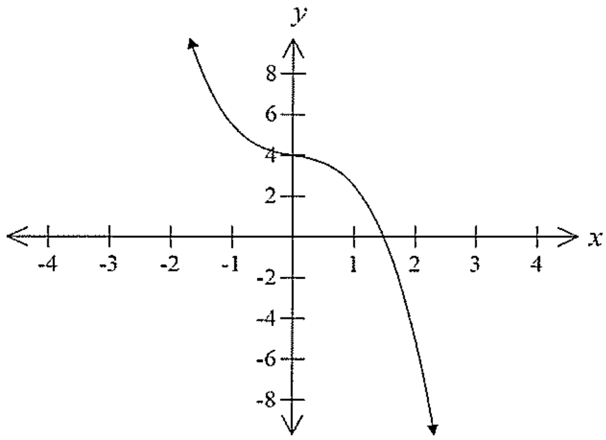
Section I	Total 10	Marks
Q1-Q10		
Section II	Total 90	Marks
Q11	/15	
Q12	/15	
Q13	/15	
Q14	/15	
Q15	/15	
Q16	/15	
	Percent	

Section I 10 marks

Attempt questions 1-10 Allow 15 minutes for this section

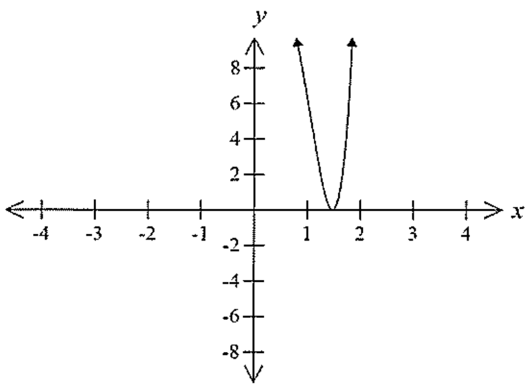
Circle the correct response on the paper below.

1 The diagram below shows the graph of the function $y = f(x)$.

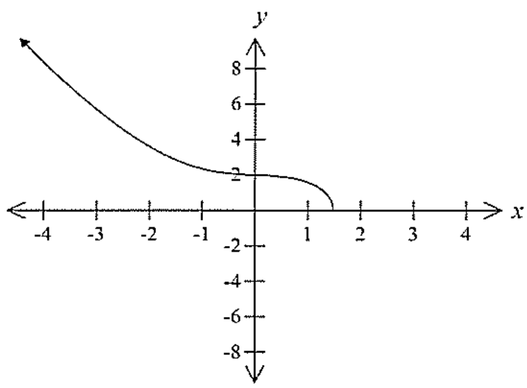


Which diagram represents the graph of $y^2 = f(x)$?

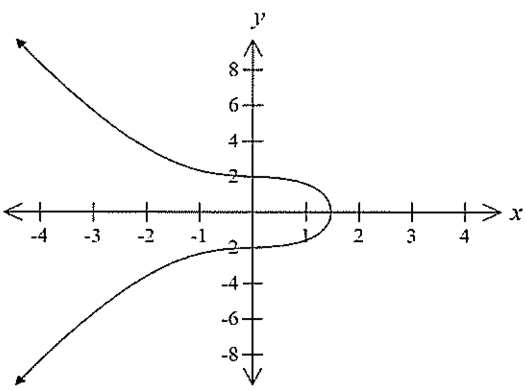
(A)



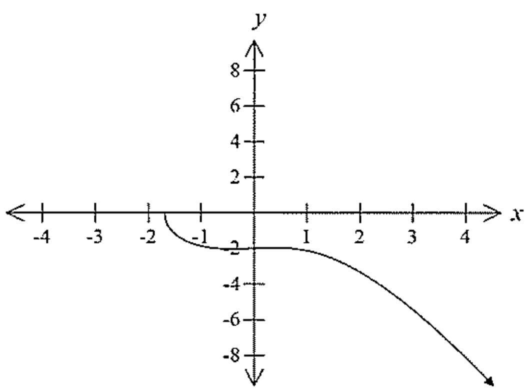
(B)



(C)



(D)



2

What is the value of $\frac{z_1}{z_2}$ given the complex numbers $z_1 = -2+2i$ and $z_2 = 1+i\sqrt{3}$?

(A) $\frac{\sqrt{3}-1}{2}+\frac{\sqrt{3}+1}{2}i$

(B) $\frac{1-\sqrt{3}}{2}-\frac{\sqrt{3}+1}{2}i$

(C) $\frac{\sqrt{3}-1}{4}+\frac{\sqrt{3}+1}{4}i$

(D) $\frac{1-\sqrt{3}}{4}-\frac{\sqrt{3}+1}{4}i$

3

For the ellipse with the equation $\frac{x^2}{4}+\frac{y^2}{3}=1$. What is the eccentricity?

(A) $\frac{1}{4}$

(B) $\frac{1}{2}$

(C) $\frac{3}{4}$

(D) $\frac{9}{16}$

4

Consider the hyperbola with the equation $\frac{x^2}{144}-\frac{y^2}{25}=1$.

What are the equations of the directrices?

(A) $x=\pm\frac{13}{144}$

(B) $x=\pm\frac{13}{25}$

(C) $x=\pm\frac{25}{13}$

(D) $x=\pm\frac{144}{13}$

5

What is the value of $\int_0^1 \frac{e^x}{1+e^x}\,dx$?

(A) $\log_e(1+e)$

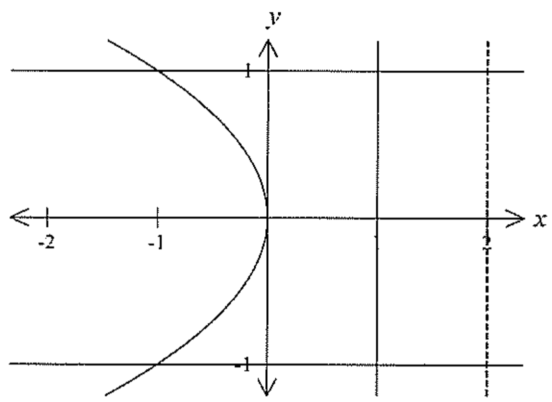
(B) 1

(C) $\log_e \frac{(1+e)}{2}$

(D) $\log_e \frac{e}{2} - 2$

6

The region is bounded by the lines $x = 1$, $y = 1$, $y = -1$ and by the curve $x = -y^2$. The region is rotated through 360° about the line $x = 2$ to form a solid. What is the correct expression for volume of this solid?



(A) $V = \int_{-1}^{1} \pi\left(y^4 - 4y^2 + 3\right)dy$

(B) $V = \int_{-1}^{1} \pi\left(y^4 + 4y^2 + 3\right)dy$

(C) $V = \int_{-1}^{1} \pi\left(y^4 - 4y^2 + 4\right)dy$

(D) $V = \int_{-1}^{1} \pi\left(y^4 + 4y^2 + 4\right)dy$

7

A particle of mass m falls from rest under gravity and the resistance to its motion is mkv^2, where v is its speed and k is a positive constant. Which of the following is the correct expression for square of the velocity where x is the distance fallen?

(A) $v^2 = \frac{g}{k}\left(1 - e^{-2kx}\right)$

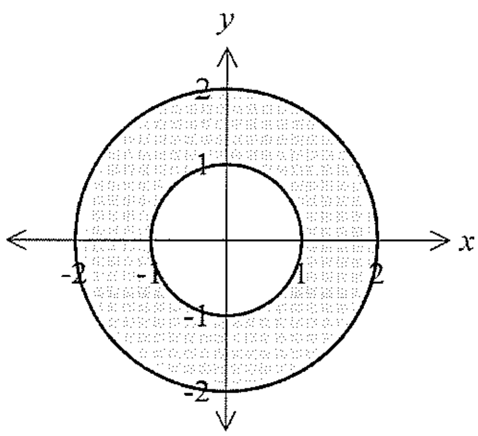
(B) $v^2 = \frac{g}{k}\left(1 + e^{-2kx}\right)$

(C) $v^2 = \frac{g}{k}\left(1 - e^{2kx}\right)$

(D) $v^2 = \frac{g}{k}\left(1 + e^{2kx}\right)$

8

Consider the Argand diagram below.



Which inequality could define the shaded area?

(A) $0 \le |z| \le 2$

(B) $1 \le |z| \le 2$

(C) $0 \le |z-1| \le 2$

(D) $1 \le |z-1| \le 2$

9

The points $P(cp, \frac{c}{p})$ and $Q(cq, \frac{c}{q})$ lie on the same branch of the hyperbola $xy = c^2$ $(p \neq q)$.

The tangents at P and Q meet at the point T. What is the equation of the normal to the hyperbola at P?

(A) $p^2x - py + c - cp^4 = 0$

(B) $p^3x - py + c - cp^4 = 0$

(C) $x + p^2y - 2c = 0$

(D) $x + p^2y - 2cp = 0$

10

Which of the following is an expression for $\int \frac{\sin x \cos x}{4 + \sin x} dx$?

Use the substitution $u = 4 + \sin x$.

(A) $-4\ln|4+\sin x| + c$

(B) $4\ln|4+\sin x| + c$

(C) $-\sin x - 4\ln|4+\sin x| + c$

(D) $4 + \sin x - 4\ln|4+\sin x| + c$

Section II

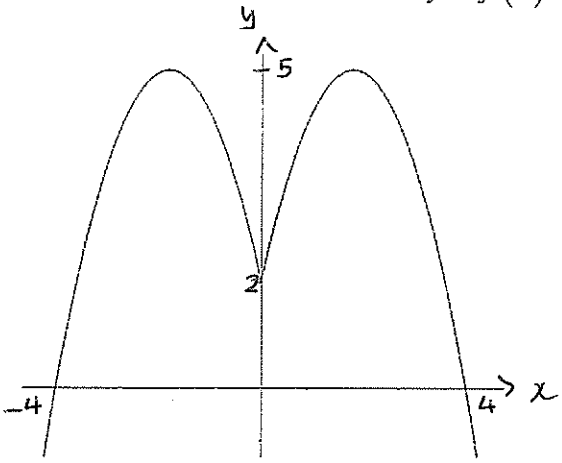
90 marks

Attempt Questions 11-16

Allow about 2 hours and 45 minutes for this section

Question 11 *(15 marks)* *Marks*

a. The sketch is of the even function $y = f(x)$



On separate number planes, sketch each of the following. Clearly showing important features

i. $y = f(x) - 2$ *1*

ii. $y = f(x-2)$ *1*

iii. $y = |f(x)|$ *1*

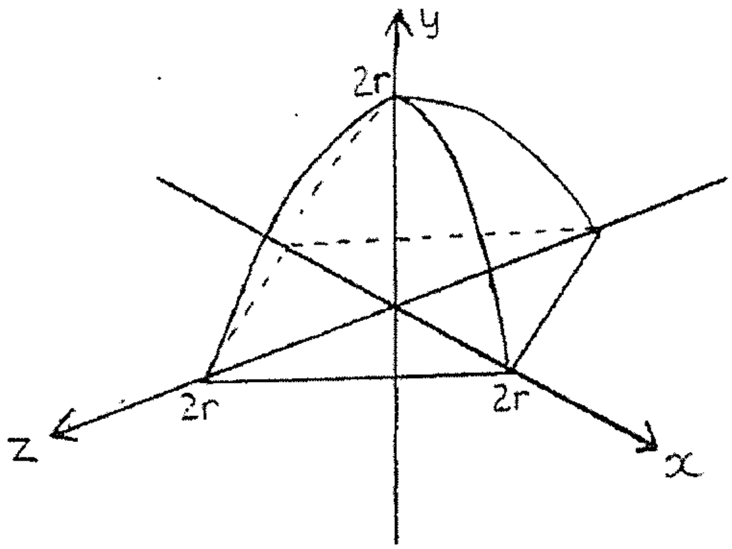
iv. $y^2 = f(x)$ *2*

v. $y = \dfrac{1}{f(x)}$ *2*

b.

Find $\displaystyle\int_{\sqrt{2}+1}^{3} \frac{dx}{\sqrt{3+2x-x^2}}$ *3*

c. The solid shown stands on a square base and the cross sections parallel to the base are squares with the diagonal being chords of a circle with centre at the origin and radius $2r$ units. 5



Show that the volume is $\dfrac{32r^3}{3}$ units cubed.

Question 12 *(15 marks)* *Marks*

a. Let $z = 3 - 2i$ and $u = -5 + 6i$

i. Find $\operatorname{Im}(uz)$ *1*

ii. Find $|u - z|$ *1*

iii. Find $\overline{-2iz}$ *1*

iv. Express $\dfrac{u}{z}$ in the form $a + bi$, where a and b are real numbers. *1*

b.

If $2+i$ is a root of $P(x) = x^4 - 6x^3 + 9x^2 + 6x - 20$, resolve $P(x)$ into irreducible factors over the field of complex numbers. *4*

c.

i. Sketch the hyperbola $x = 4\sec\theta$, $y = 3\tan\theta$ showing clearly any points of intersection with the axes, the coordinates of the foci, the equation of the directrices and the equation of the asymptotes. *4*

ii. If $P(x_1, y_1)$ is any point on the hyperbola in part i, show that $|PS - PS'| = 8$ where S and S' are the foci of the hyperbola. *3*

Question 13 *(15 marks)* *Marks*

a. Suppose that $f\left(x\right)$ is the function: *3*

$$f\left(x\right)=\begin{cases}\frac{1}{4}(4+x)(2-x), \mathit{for}\ x<0 \\ \frac{1}{4}(4-x)(2+x), \mathit{for}\ x>0\end{cases}$$

Sketch on a number plane the graph of the function $y=f'\left(x\right)$, showing all the important features.

b.

i. Evaluate $\int_0^{\frac{\pi}{4}} \frac{\sec^2 x}{1+\tan x}dx$ *3*

ii. Find an expression for $\int_0^{\ln x} e^x \sin\left(e^x\right)dx$, in its simplest form *3*

c.

i. If α, β and γ are the roots of the cubic equation $x^3-3x^2-6x+7=0$ find, the equation whose roots are α^2, β^2 and γ^2 *3*

i. The roots of the equation $t^3+qt-r=0$ are *a, b* and *c*.
If $S_n=a^n+b^n+c^n$ where n is a positive integer, prove that *3*
$S_{n+3}=r\,S_n-q\,S_{n+1}$

Question 14 *(15 marks)* *Marks*

a.

i. Using DeMoivre's theorem show that the solutions of the equation $z^3 = 1$ in the complex number system are : 2

$$z = \cos\theta + i\sin\theta \quad \text{for} \quad \theta = 0, \frac{2\pi}{3}, \frac{4\pi}{3}$$

ii. If $\omega = cis\dfrac{2\pi}{3}$ show that $\omega^2 + \omega + 1 = 0$ and $\omega^3 - \omega^2 - \omega - 2 = 0$ 2

iii. Hence or otherwise solve the cubic equation $z^3 - z^2 - z - 2 = 0$ 3

b.

i. Find the equation of the tangent and the normal to the ellipse $x^2 + 4y^2 = 100$ at the point $P(8,-3)$. 4

ii. The normal at P meets the major axis of the ellipse at G. The perpendicular from the centre to the tangent at P meets this tangent at K. Show that $PG \times OK$ is equal to the square of the semi-minor axis of the ellipse. 4

Question 15

Marks

a. A particle is fired vertically upwards with initial velocity V metres per second, and is subject both to constant gravity, and to air resistance proportional to speed, so that its equation of motion is: $\ddot{x} = -g - kv$, where $k > 0$ is a constant, and g is acceleration due to gravity. 5

By replacing $\ddot{x}$ by $v\dfrac{dv}{dx}$ and integrating, prove that the projectile reaches a maximum height H given by:

$$H = \frac{V}{k} - \frac{g}{k^2}\ln\left(1 + \frac{kV}{g}\right)$$

b.

On separate Argand diagrams sketch:

i. $|z - 2i| < 2$ 2

ii. $\arg\left(z - (1+i)\right) = \dfrac{3\pi}{4}$ 2

c.

i. Show that $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$ 2

ii. Hence show that $\int_0^{\frac{\pi}{4}} \ln(1+\tan x)\,dx = \int_0^{\frac{\pi}{4}} \ln\left(\dfrac{2}{1+\tan x}\right) dx$ 2

iii. Hence evaluate $\int_0^{\frac{\pi}{4}} \ln(1+\tan x)\; dx$ 2

Question 16

a.

Marks

Area enclosed by $y=(x-2)^2$ and the line $y=4$ is rotated about the y axis.

i. Draw a diagram to illustrate this.

ii. Using the cylindrical shells find the volume of the solid formed. *5*

b.

i. If a is a multiple root of the polynomial $P(x)=0$, prove that $P'(a)=0$. *2*

ii. Find all the roots of the equation $16x^3-12x^2+1=0$ given that two of the roots are equal. *3*

c.

A weight is oscillating on the end of a spring under water. Because of the resistance by the water (proportional to speed), the equation of the particle is : $\ddot{x}=-4x-2\sqrt{3}\,\dot{x}$ .where x is the distance in metres above equilibrium position at time t seconds. Initially the particle is at the equilibrium position, moving upwards with a speed of $3\ m/s$

i. Find the first and second derivatives of $x=Ae^{-\sqrt{3}t}\sin t$, where A is the constant, and hence show that $x=Ae^{-\sqrt{3}t}\sin t$, is a solution of the differential equation, $\ddot{x}=-4x-2\sqrt{3}\,\dot{x}$,then substitute the initial conditions to find A. *3*

ii. At what times during the first 2π seconds is the particle moving downwards? *2*

End of examination

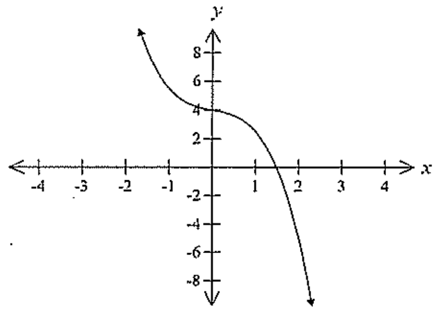
Mathematics Extension 2 trial 2015 – Solutions

Section I 10 marks

Attempt questions 1-10 Allow 15 minutes for this section

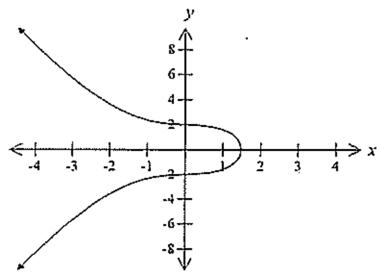
Circle the correct response on the paper below.

1 The diagram below shows the graph of the function $y = f(x)$.



Which diagram represents the graph of $y^2 = f(x)$?

(C)



2

What is the value of $\frac{z_1}{z_2}$ given the complex numbers $z_1 = -2+2i$ and $z_2 = 1+i\sqrt{3}$?

$$\frac{z_1}{z_2} = \frac{-2+2i}{1+i\sqrt{3}} \times \frac{1-i\sqrt{3}}{1-i\sqrt{3}}$$
$$= \frac{-2+2i\sqrt{3}+2i+2\sqrt{3}}{1+3}$$
$$= \frac{2(-1+\sqrt{3})+i(1+\sqrt{3})}{4}$$
$$= \frac{\sqrt{3}-1}{2} + \frac{\sqrt{3}+1}{2}i$$

(A) $\frac{\sqrt{3}-1}{2} + \frac{\sqrt{3}+1}{2}i$

3

For the ellipse with the equation $\frac{x^2}{4} + \frac{y^2}{3} = 1$. What is the eccentricity?

$$b^2 = a^2(1-e^2)$$
$$3 = 4(1-e^2)$$
$$(1-e^2) = \frac{3}{4} \text{ or } e^2 = \frac{1}{4} \text{ or } e = \frac{1}{2}$$

(B) $\frac{1}{2}$

4

Consider the hyperbola with the equation $\frac{x^2}{144}-\frac{y^2}{25}=1$.

What are the equations of the directrices?

$b^2=a^2(e^2-1)$ $\quad$ $a^2=144$ and $b^2=25$.

$25=144(e^2-1)$ $\quad$ $a=12$ $\quad$ $b=5$

$(e^2-1)=\frac{25}{144}$ or $e^2=\frac{169}{144}$ or $e=\frac{13}{12}$

Equation of the directrices are $x=\pm\frac{a}{e}=\pm\frac{144}{13}$.

(D) $x=\pm\frac{144}{13}$

5

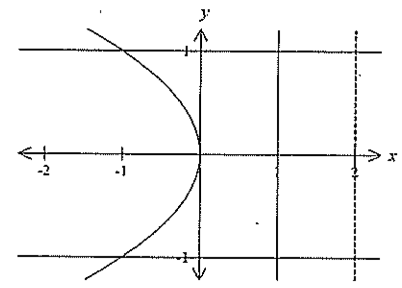
What is the value of $\int_0^1 \frac{e^x}{1+e^x}dx$?

$$\int_0^1 \frac{e^x}{1+e^x}dx=\left[\log_e\left(1+e^x\right)\right]_0^1$$
$$=\log_e(1+e)-\log_e 2$$
$$=\log_e\frac{(1+e)}{2}$$

(C) $\log_e\frac{(1+e)}{2}$

6

The region is bounded by the lines $x=1$, $y=1$, $y=-1$ and by the curve $x=-y^2$. The region is rotated through 360° about the line $x=2$ to form a solid. What is the correct expression for volume of this solid?



Area of the slice is an annulus

Inner radius is 1 and outer radius is $2+y^2$ and height y

$$A=\pi\left(R^2-r^2\right)$$
$$=\pi\left((2+y^2)^2-1^2\right)$$
$$=\pi\left(4+4y^2+y^4-1\right)$$
$$=\pi\left(y^4+4y^2+3\right)$$
$$\delta V=\delta A.\delta y$$
$$V=\lim_{\delta y\to 0}\sum_{y=-1}^{1}\pi\left(y^4+4y^2+3\right)\delta y$$
$$=\int_{-1}^{1}\pi\left(y^4+4y^2+3\right)dy$$

(B) $V=\int_{-1}^{1}\pi\left(y^4+4y^2+3\right)dy$

7

A particle of mass m falls from rest under gravity and the resistance to its motion is mkv^2, where v is its speed and k is a positive constant. Which of the following is the correct expression for square of the velocity where x is the distance fallen?

$$\dot{v} = g - kv^2$$

$$\frac{1}{2}\frac{dv^2}{dx} = g - kv^2$$

$$2\,dx = \frac{dv^2}{g - kv^2}$$

$$-2k\,dx = \frac{-k\,dv^2}{g - kv^2}$$

$$-2kx + c = \log_e\left|g - kv^2\right|$$

Initial conditions t = 0, v = 0 and $x = 0$ or $c = \log_e g$

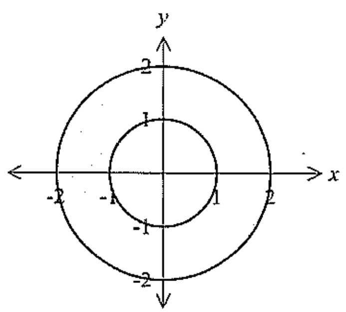
$$-2kx = \log_e\left|1 - \frac{k}{g}v^2\right|$$

$$v^2 = \frac{g}{k}\left(1 - e^{-2kx}\right)$$

(A) $v^2 = \frac{g}{k}\left(1 - e^{-2kx}\right)$

8

Consider the Argand diagram below.



Which inequality could define the shaded area?

$|z| \leq 1$ represents a region with a centre is (0, 0) and radius is greater than or equal to 1.

$|z| \leq 2$ represents a region with a centre is (0, 0) and radius is less than or equal to 1.

$1 \leq |z| \leq 2$

(B) $1 \leq |z| \leq 2$

9

The points $P(cp, \frac{c}{p})$ and $Q(cq, \frac{c}{q})$ lie on the same branch of the hyperbola $xy = c^2$ $(p \neq q)$.

The tangents at P and Q meet at the point T. What is the equation of the normal to the hyperbola at P?

To find the gradient of the tangent.

$$xy = c^2$$

$$x\frac{dy}{dx} + y = 0$$

$$\frac{dy}{dx} = -\frac{y}{x}$$

At P $(cp, \frac{c}{p})$ $\frac{dy}{dx} = -\frac{\frac{c}{p}}{cp} = -\frac{1}{p^2}$

Gradient of the normal is p^2 $(m_1 m_2 = -1)$

Equation of the normal at P $(cp, \frac{c}{p})$

$$y - \frac{c}{p} = p^2(x - cp)$$

$$py - c = p^3x - cp^4$$

$$p^3x - py + c - cp^4 = 0$$

(B) $p^3x - py + c - cp^4 = 0$

10

Which of the following is an expression for $\int \frac{\sin x \cos x}{4+\sin x}\,dx$?

Use the substitution $u = 4 + \sin x$.

Let $u = 4 + \sin x$ then

$\frac{du}{dx} = \cos x$

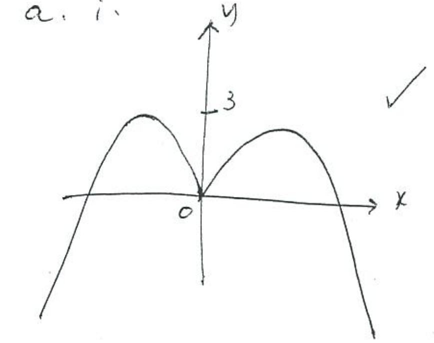
Now $\sin x = u - 4$

$$\begin{aligned}\int \frac{\sin x \cos x}{4+\sin x}\,dx &= \int \frac{(u-4)\cos x}{u}\,\frac{du}{\cos x} \\ &= \int 1 - \frac{4}{u}\,du \\ &= u - 4\ln|u| + c \\ &= 4 + \sin x - 4\ln|4+\sin x| + c \\ &= \sin x - 4\ln|4+\sin x| + c\end{aligned}$$

(D) $4 + \sin x - 4\ln|4+\sin x| + c$

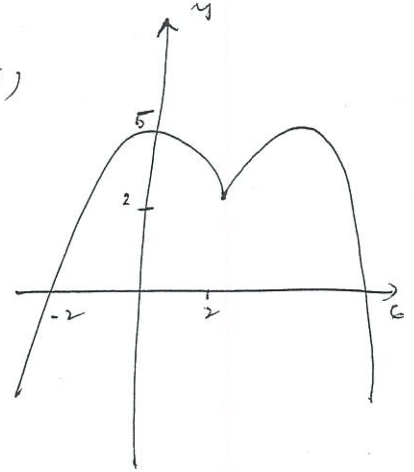
Question 11

a. i.



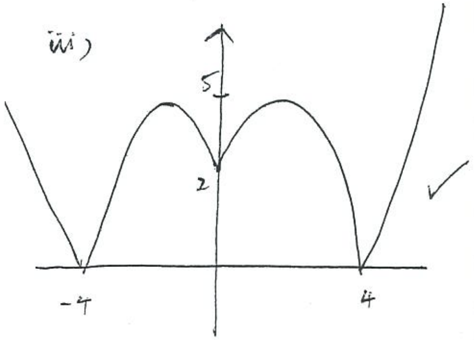
✓

ii)



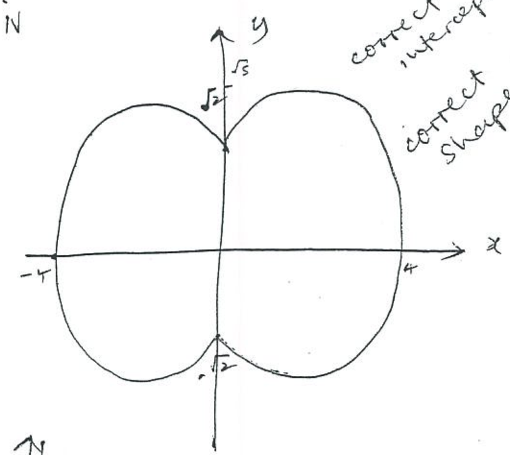
✓

iii)



✓

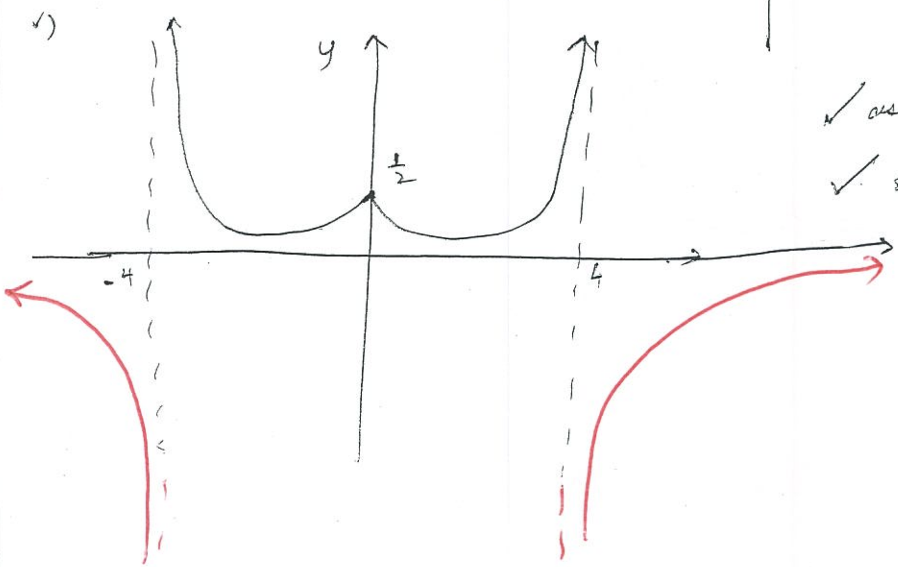
iv



correct intercepts ✓

correct shape ✓

v)



✓ asymptotes

✓ shape

b) $\displaystyle\int_{\sqrt{2}+1}^{3} \frac{dx}{\sqrt{3+2x-x^2}}$

$-x^2+2x+3$

$-(x^2-2x)+3$

$-(x^2-2x+1)+3+1$

$-(x-1)^2+4$ ✓

$\displaystyle\int_{\sqrt{2}+1}^{3} \frac{dx}{\sqrt{2^2-(x-1)^2}}$

$= \sin^{-1} \frac{x-1}{2} \Big]_{\sqrt{2}+1}^{3}$ ✓

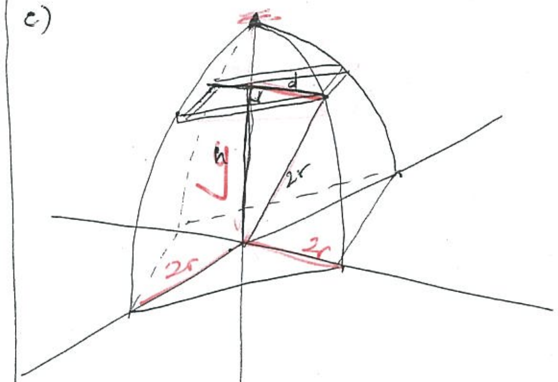
$= \sin^{-1}\frac{3-1}{2} - \sin^{-1}\frac{\sqrt{2}+1-1}{2}$

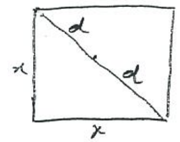
$= \sin^{-1} 1 - \sin^{-1}\frac{1}{\sqrt{2}}$

$= \frac{\pi}{2} - \frac{\pi}{4}$

$= \frac{\pi}{4}$ ✓

c)





$d = \sqrt{(2r)^2 - y^2}$

$d = \sqrt{4r^2 - y^2}$ ✓

$x^2 + x^2 = (2d)^2$

$2x^2 = 2d^2$

$2x^2 = \cancel{2}4(4r^2 - y^2)$

✓ $x^2 = 2(4r^2 - y)$

Area of squa

$V \doteq \lim_{dh \to 0} \sum_{0}^{2r} 2(4r^2 - y^2)\, dy$ ✓

$= \int_0^{2r} 2(4r^2 - y^2)\, dy$ ✓

$= 2\left[4r^2 y - \frac{y^3}{3}\right]_0^{2r}$ ✓

$= 2\left[4r^2(2r) - \frac{(2r)^3}{3}\right]$

$= 2\left[8r^3 - \frac{8r^3}{3}\right]$

$V = \frac{32}{8} r^3 \quad u^3$

Question 11.

a) Mostly well done

- students did not change x and y intercepts
-

(b) Mostly well done

- some students tried to do this as a log question

(c) The question said to use cross sections perpendicular to the y axis. Some students used dx.

. students found the area of the base

. Students integrated with respect to R, but R is a constant.

. Some students just fudged the answers to get $\frac{32}{8}\pi r^3$.

Question 12

$z = 3-2i \qquad u = -5+6i$

$\text{Im}(uz)$

$uz = (-5+6i)(3-2i)$

$= -3+28i$

$\text{Im}(uz) = 28$ ✓

Well done

$u-z = (-5+6i) - (3-2i)$

$= -8+8i$

$|u-z| = \sqrt{(-8)^2+(8)^2}$

$= \sqrt{128}$

$= 8\sqrt{2}$ ✓

Well done

$-2\overline{iz}$

$iz = i(3-2i)$

$= 2+3i$

$-2(iz) = -2(2+3i)$

$= -4-6i$

$-2\overline{iz} = -4+6i$ ✓

A few errors with +/- signs

$\frac{u}{z} = \frac{-5+6i}{3-2i} \times \frac{3+2i}{3+2i}$

$= \frac{-15-10i+18i+12i^2}{9+4}$

$= \frac{-27+8i}{13}$

$= -\frac{27}{13} + \frac{8}{13}i$ ✓

Well done

(5) $2+i$ is a root

$\therefore 2-i$ is a root ✓

$[x-(2+i)][x-(2-i)]$

$[x^2-(2+i)x-(2-i)x+(2+i)(2-i)]$

$[x^2-4x+5]$ ✓

$$\begin{array}{r}
x^2-2x-4 \\
x^2-4x+5 \,\big)\, x^4-6x^3+9x^2+6x-20 \\
-(x^4-4x^3+5x^2) \\
\hline
-2x^3+4x^2+6x-20 \\
-(-2x^3+8x^2-10x) \\
\hline
-4x^2+16x-20 \\
-(-4x^2+16x-20) \\
\hline
\end{array}$$

$P(x) = (x-(2+i))(x-(2-i))(x^2-2x-4)$ ✓

$x^2-2x-4=0$

$x = \frac{2 \pm \sqrt{4-4(-4)}}{2}$

$x = 1 \pm \frac{\sqrt{20}}{2}$

$x = 1 \pm \sqrt{5}$

$P(x) = (x-(2+i))(x-(2-i))(x-(1+\sqrt{5}))(x-(1-\sqrt{5}))$ ✓

Many students neglected to break down x^2-2x-4 into irreducible factors. Some students made errors using quadratic formula when attempting this.

(c) $x = 4\sec\theta \quad y = 3\tan\theta$

$\tan^2\theta + 1 = \sec^2\theta$

$\left(\frac{y}{3}\right)^2 + 1 = \left(\frac{x}{4}\right)^2$

$\frac{x^2}{16} - \frac{y^2}{9} = 1$

$b^2 = a^2(e^2 - 1)$

$e^2 = 1 + \frac{9}{16}$

$e^2 = \frac{25}{16}$

$e = \frac{5}{4}$ ✓

foci $(\pm ae, 0)$

$(4 \times \pm\frac{5}{4}, 0)$

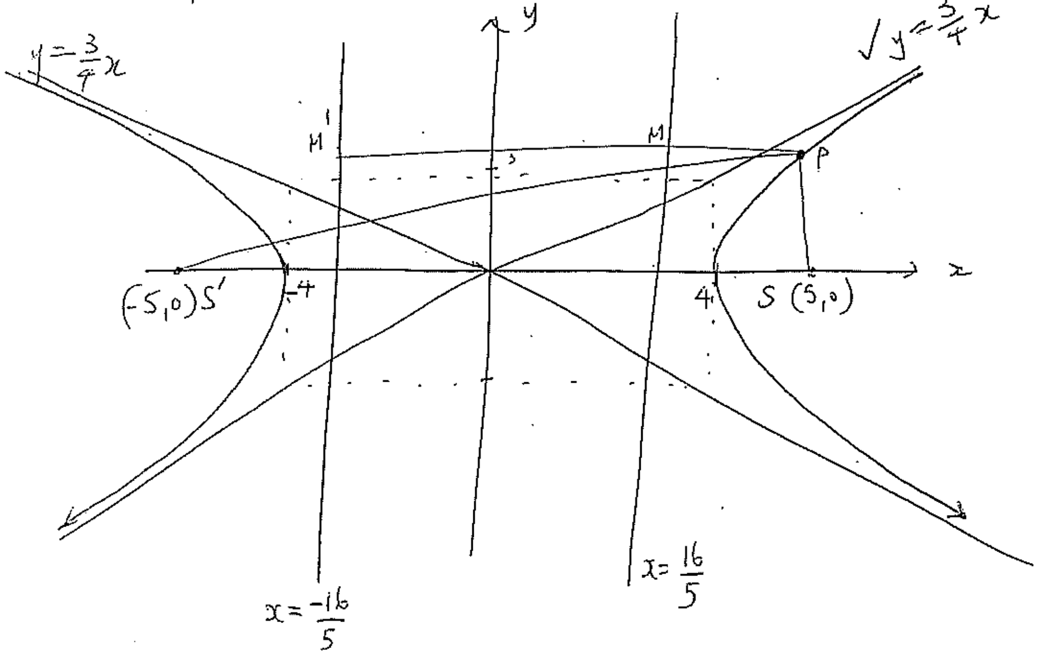
$(\pm 5, 0)$ ✓

directrices

$x = \pm \frac{a}{e}$

$x = \pm \frac{4}{5/4}$

$x = \pm \frac{16}{5}$ ✓



Well done.

c ii) $\frac{PS}{PM} = e$ ✓ $\quad \frac{PS'}{PM'} = e$

$PS = e\,PM \quad PS' = e\,PM'$

$LHS = |PS - PS'|$ ✓

$= |e \cdot PM - e \cdot PM'|$

$= e\,|PM - PM'|$

$= e\left|\left(x - \frac{16}{5}\right) - \left(x + \frac{16}{5}\right)\right|$

$= \frac{5}{4}\left|-\frac{32}{5}\right|$ ✓

$= 8$

Many students didn't define M and M' on a diagram.

Skipping steps of the proof was also a problem.

12 a i) Well done

ii) Well done

iii) A few errors with +/− signs

iv) Well done.

12 b Many students neglected to break down $x^2 - 2x - 4$ into irreducible (i.e linear) factors. Some students made errors using quadratic formula when attempting this.

12 c i Well done

12 c ii Many students didn't define M and M' on a diagram.
Skipping steps of the proof was also a problem.

Question 13

a)

$f(x) = \frac{1}{4}(4+x)(2-x) \quad , x<0$

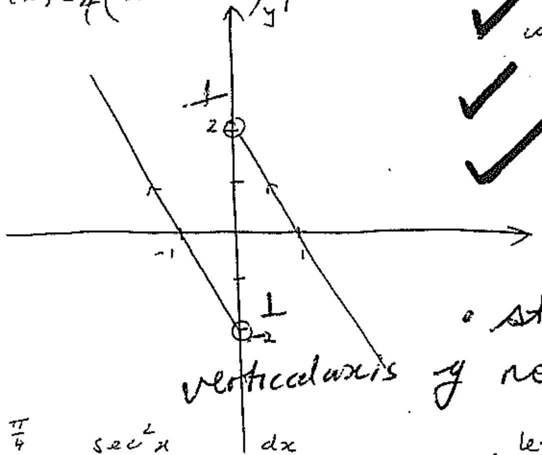
$= \frac{1}{4}(8-2x-x^2)$

$f'(x) = \frac{1}{4}(-2-2x)$

$f(x) = \frac{1}{4}(4-x)(2+x) \quad , x>0$

$= \frac{1}{4}(8+2x-x^2)$

$f'(x) = \frac{1}{4}(2-2x)$



✓ correct shape

✓ $x \neq 0$

✓ intercepts

• students labelled vertical axis y not y'

ⓑ i $\int_0^{\frac{\pi}{4}} \frac{\sec^2 x}{1+\tan x}\,dx$

$= \int_1^2 \frac{\cancel{\sec^2 x}}{u} \cdot \frac{du}{\cancel{\sec^2 x}}$ ✓

$= \ln \Big]_1^2$

$= \ln 2$ ✓

✓ let $u = 1 + \tan x$

$\frac{du}{dx} = \sec^2 x$

$dx = \frac{du}{\sec^2 x}$

when $x = \frac{\pi}{4}$ $\quad u = 1 + \tan\frac{\pi}{4}$

$u = 2$

$x = 0 \quad u = 1 + \tan 0$

$u = 1.$

well done

ⓑ ii $\int_0^{\ln x} e^x \sin(e^x)\,dx$

when $x = \ln x$ $\quad u = e^{(\ln x)}$

$u = x$

$x = 0 \quad u = e^0 = 1$

$= \int_1^x \cancel{e^x} \sin u \cdot \frac{du}{\cancel{e^x}}$ ✓

let $u = e^x$

$\frac{du}{dx} = e^x$ ✓

$dx = \frac{du}{e^x}$

$= -\cos u \Big]_1^x$ ✓

$= -\cos x + \cos 1$

well done

ⓒ i) α, β, γ are the roots of $x^3 - 3x^2 - 6x + 7 = 0$

let $x = \alpha^2$ ✓

$\alpha = \sqrt{x}$

$(\sqrt{x})^3 - 3(\sqrt{x})^2 - 6(\sqrt{x}) + 7 = 0$

$x\sqrt{x} - 3x - 6\sqrt{x} + 7 = 0$

$x\sqrt{x} - 6\sqrt{x} = 3x - 7$ ✓

$(x\sqrt{x} - 6\sqrt{x})^2 = (3x-7)^2$

$(\sqrt{x}(x-6))^2 = (3x-7)^2$

$x(x^2 - 12x + 36) = 9x^2 - 42x + 49$

$x^3 - 12x^2 + 36x - 9x^2 + 42x - 49 = 0$ ✓

$x^3 - 21x^2 + 78x - 49 = 0$

very well done

(c) ii)

$t^3 + qt - r = 0$

$S_n = a^n + b^n + c^n$

$S_{n+3} = a^{n+3} + b^{n+3} + c^{n+3}$

$= a^3 \cdot a^n + b^3 b^n + c^3 c^n$ ✓

$S_{n+1} = a \cdot a^n + b \cdot b^n + c \cdot c^n$

$a^3 + q \cdot a - r = 0$

$a^3 = r - qa$
$b^3 = r - qb$
$c^3 = r - qc$ ✓

$S_{n+3} = a^3 \cdot a^n + b^3 \cdot b^n + c^3 \cdot c^n$

$= (r - qa)\, a^n + (r - qb)\, b^n + (r - qc) \cdot c^n$

$= r \cdot a^n - q \cdot a \cdot a^n + r b^n - q \cdot b \cdot b^n + r c^n - q c \cdot c^n$

$= r(a^n + b^n + c^n) - q(a^{n+1} + b^{n+1} + c^{n+1})$ ✓

$= r S_n - q S_{n+1}$

$S_{n+3} = r S_n - q S_{n+1}$

many students began at RHS and tried to show by expansion. This took a lot longer.

Question 13

a) Very badly done

- many students did not graph $f'(x)$.
- $f'(x)$ was given as a curve
- students included $x=0$ in their answers

(b) i) Well done.

ii) - students got into trouble trying to solve this using integration by parts.

- students used a substitution but did not change int x values into u values.

(c) i) Very well done.

ii) - Many students began with the right hand side and this became very algebra heavy.

i)

$z = \cos\theta + i\sin\theta$

$z^3 = r^3(\cos\theta + i\sin\theta)^3$

$z^3 = r^3(\cos 3\theta + i\sin 3\theta)$

$z^3 = 1$ $\quad r$

$|z^3| = \sqrt{1^2 + 0^2}$

$|z| = 1$

$r = 1$.

$\arg z^3 = 3\theta$

$\arg z^3 = 3 \cdot \arg z$

$3\theta = 0, 2\pi, 4\pi, \ldots$

$\theta = 0, \frac{2\pi}{3}, \frac{4\pi}{3}, \ldots$

$z_1 = \text{cis } 0$

$z_2 = \text{cis } \frac{2\pi}{3}$

$z_3 = \text{cis } \frac{4\pi}{3}$

ii) $\omega = \text{cis } \frac{2\pi}{3}$

If ω is a root then

$\omega^3 = 1$.

$\omega^3 - 1 = 0$

$(\omega - 1)(\omega^2 + \omega + 1) = 0 \leftarrow$ ✓

$\therefore \omega^2 + \omega + 1 = 0$

$\omega^2 + \omega + 1 = 0$

$-\omega^2 - \omega - 1 = 0$

$\omega^3 - \omega^2 - \omega - 1 = 1 \leftarrow$ ✓

$\omega^3 - \omega^2 - \omega - 2 = 0$

iii) $z^3 - z^2 - z - 2 = 0$

$P(z) = z^3 - z^2 - z - 2$, let $z = 2$

$P(2) = (2)^3 - (2)^2 - (2) - 2$ ✓

$P(2) = 0$

$\therefore z = 2$ is a root

$$\begin{array}{r|l} & z^2 + z + 1 \\ \hline z-2 & z^3 - z^2 - z - 2 \\ & -(z^3 - 2z^2) \\ \hline & z^2 - z - 2 \\ & -(z^2 - 2z) \\ \hline & z - 2 \\ & -(z-2) \\ \hline & 1 \end{array}$$

✓

$P(z) = (z-2)(z^2 + z + 1)$

$z^2 + z + 1 = 0$

$z = \frac{-1 \pm \sqrt{1-4}}{2}$

$z = \frac{-1}{2} \pm \frac{\sqrt{3}}{2} i$

$z = -\frac{1}{2} + \frac{\sqrt{3}}{2} i, -\frac{1}{2} - \frac{\sqrt{3}}{2} i$ ✓

(b) i) $x^2 + 4y^2 = 100$

$$\frac{d}{dx}(x^2) + \frac{d}{dx}(4y^2) = \frac{d}{dx}(100)$$

$$2x + 8y \cdot \frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = -\frac{2x}{8y}.$$

$$\frac{dy}{dx} = -\frac{x}{4y}$$

at $(8,-3)$ $\quad \frac{dy}{dx} = -\frac{8}{4(-3)}$

$$\frac{dy}{dx} = \frac{8}{12} = \frac{2}{3}$$

equation of the tangent

$$(y-(-3)) = \frac{2}{3}(x-8)$$

$$y+3 = \frac{2}{3}(x-8)$$

$$3y+9 = 2x-16$$

$$2x-3y-25=0$$

equation of the normal

$$(y-(-3)) = -\frac{3}{2}(x-8)$$

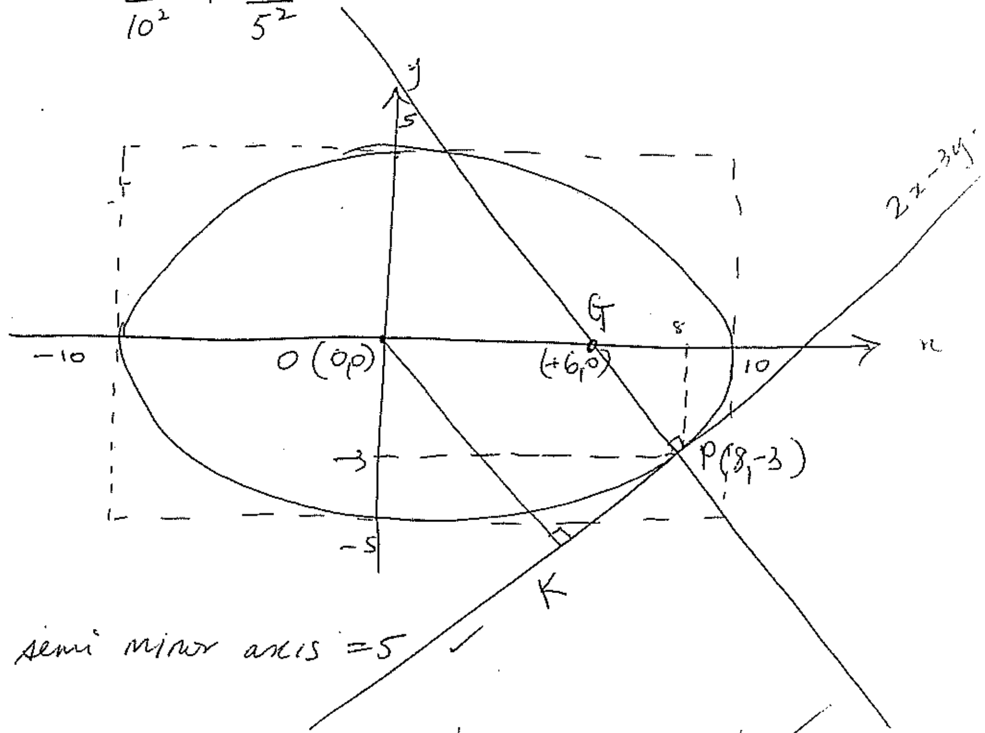
$$y+3 = -\frac{3}{2}(x-8)$$

$$2y+6 = -3x+24$$

$$3x+2y-18=0$$

6 ii $x^2+4y^2=100$

$$\frac{x^2}{10^2} + \frac{y^2}{5^2} = 1.$$



semi minor axis $=5$

Perp. distance OK $= \frac{|2(0)-3(0)-25|}{\sqrt{2^2+3^2}}$

$$= \frac{25}{\sqrt{13}}$$

Perp. distance PG $= \frac{|2(+6)-3(0)-25|}{\sqrt{2^2+3^2}}$

$$= \frac{|+12-25|}{\sqrt{13}}$$

$$= \frac{13}{\sqrt{13}}$$

$PG \times CK = \frac{25}{\sqrt{13}} \times \sqrt{13} = 25 = 5^2 =$ (semi-minor axis)^2

QUESTION 14

(A) (i) * Many students stated the roots are

$$z_n = \text{cis}\,\frac{2k\pi}{n} \quad \text{where } n = 0, 1, 2$$

without explaining how this result is derived (i.e the students must reference De Moivre's theorem).

* Many students did not show that $|z| = 1$.

* Some students incorrectly stated

$$z = 0, \frac{2\pi}{3}, \frac{4\pi}{3}$$

* Greater care should be exercised and student should be more explicit with show that questions.

(ii) * Many different approaches which were generally answered well.

* If factoring

$$\omega^3 - 1 = 0$$
$$(\omega - 1)(\omega^2 + \omega + 1) = 0$$

explain why $\omega \neq 1$ (i.e ω is given as a complex root)

(iii) * Many student gave ~~th~~ a partial answer of ω and ω^2 without giving thought to the Fundemental Theorem of Algebra (eg cubics would have 3 solutions).

(B) (i) Generally well answered.

Students must be careful to label the equations as either the tangent or normal.

(ii) * Many poorly communicated answers.

In show that questions you must be explicit regardless of how easy or obvious you think the working is.

* In this question many students stated/found

$PG \times OK = 25$ without reference to the semi-minor axis eg $= 5^2 = b^2$.

Question 15

$\ddot{x} = -g - kv$

$v\dfrac{dv}{dx} = -g - kv$

$\dfrac{dv}{dx} = \dfrac{-(g+kv)}{v}$

$\dfrac{dx}{dv} = \dfrac{-v}{g+kv}$

$-k \cdot \dfrac{dx}{dv} = \dfrac{kv}{g+k}$ ✓

$-k\dfrac{dx}{dv} = 1 - \dfrac{g}{g+kv}$

$-\dfrac{k}{g} \cdot \dfrac{dx}{dv} = \dfrac{1}{g} - \dfrac{1}{g+kv}$

$-\dfrac{k^2}{g}\dfrac{dx}{dV} = \dfrac{k}{g} - \dfrac{k}{g+kv}$ ✓

$\int -\dfrac{k^2}{g}\dfrac{dx}{dv}\,dv = \int \dfrac{k}{g} - \dfrac{k}{g+kv}\,dv$

$-\dfrac{k^2}{g}x = \dfrac{k}{g}v - \ln(g+kv) + c$

initially $v = V$, $x = 0$

$0 = \dfrac{k}{g}V - \ln(g+kV) + c$

$c = \ln(g+kV) - \dfrac{kV}{g}$ ✓

Well done.

Most common errors involved mixing up +/- signs.

$-\dfrac{k^2}{g}x = \dfrac{kv}{g} - \ln(g+kv) + \ln(g+kV) - \dfrac{kV}{g}$

$-\dfrac{k^2}{g}x = \dfrac{kv - kV}{g} + \ln\left(\dfrac{g+kV}{g+kv}\right)$

$x = \dfrac{v - V}{-k} - \dfrac{g}{k^2}\ln\left(\dfrac{g+kV}{g+kv}\right)$ ✓

max height $x = H$, $v = 0$

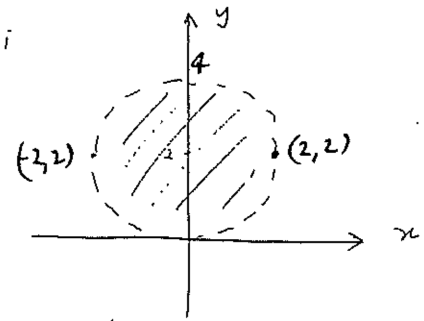
$H = \dfrac{0 - V}{-k} - \dfrac{g}{k^2}\ln\left(\dfrac{g+kV}{g+0}\right)$

$H = \dfrac{V}{k} - \dfrac{g}{k^2}\ln\left(\dfrac{g+kV}{g}\right)$

$H = \dfrac{V}{k} - \dfrac{g}{k^2}\ln\left(\dfrac{g}{g} + \dfrac{kV}{g}\right)$

$= \dfrac{V}{k} - \dfrac{g}{k^2}\ln\left(1 + \dfrac{kV}{g}\right)$ ✓

(b) i



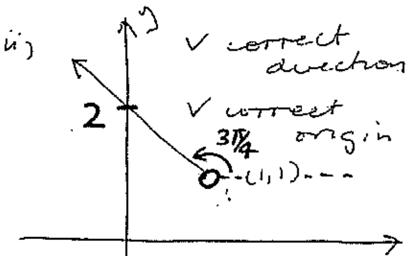
✓ correct line / shape

✓ correct shading

$z - 2i = (x+iy) - 2i$

$= x + i(y-2)$

ii)



✓ correct direction

✓ correct origin

$z - (1+i) = x + iy - 1 - i$

$= (x-1) + i(y-1)$

$\arg(z - (1+i)) = \dfrac{3\pi}{4}$

c) i) show that $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$ Well done

$LHS = \int_0^a f(x)\,dx$

let $u = a - x$.
$x = a - u$.

when $x = 0$ $u = a$
$x = a$ $u = 0$

$\frac{du}{dx} = -1$

$\therefore -du = +dx$

$= \int_a^0 -f(a-u)\,du$

$= \int_0^a f(a-u)\,du$

$= \int_0^a f(a-x)\,dx$

ii) show $\int_0^{\frac{\pi}{4}} \ln(1+\tan x)\,dx = \int_0^{\frac{\pi}{4}} \ln\left(\frac{2}{1+\tan x}\right)dx$

$LHS = \int_0^{\frac{\pi}{4}} \ln(1+\tan x)\,dx$

$\therefore = \int_0^{\frac{\pi}{4}} \ln\left(1+\tan\left(\frac{\pi}{4}-x\right)\right)dx$ from (i)

$= \int_0^{\frac{\pi}{4}} \ln\left(1 + \frac{\tan\frac{\pi}{4} - \tan x}{1+\tan\frac{\pi}{4}\tan x}\right)dx$

$= \int_0^{\frac{\pi}{4}} \ln\left(1 + \frac{1-\tan x}{1+\tan x}\right)dx$

$= \int_0^{\frac{\pi}{4}} \ln\left(\frac{1+\cancel{\tan x}+1-\cancel{\tan x}}{1+\tan x}\right)dx$

Some students skipped 2nd last step showing the common denominator.

$= \int_0^{\frac{\pi}{4}} \ln\left(\frac{2}{1+\tan x}\right)dx$

iii) $\int_0^{\frac{\pi}{4}} \ln(1+\tan x)\,dx$

$= \int_0^{\frac{\pi}{4}} \ln\left(\frac{2}{1+\tan x}\right)dx$

$= \int_0^{\frac{\pi}{4}} \ln 2 - \ln(1+\tan x)\,dx$

$\int_0^{\frac{\pi}{4}} \ln(1+\tan x)\,dx = \int_0^{\frac{\pi}{4}} \ln 2 - \int_0^{\frac{\pi}{4}} \ln(1+\tan x)\,dx$

$2\int_0^{\frac{\pi}{4}} \ln(1+\tan x)\,dx = \left[x.\ln 2\right]_0^{\frac{\pi}{4}}$

$\int_0^{\frac{\pi}{4}} \ln(1+\tan x)\,dx = \frac{1}{2}\left(\frac{\pi}{4}.\ln 2 - 0\right)$

$\int_0^{\frac{\pi}{4}} \ln(1+\tan x) = \frac{\pi}{8}\ln 2$

Many students didn't make use of 2nd log law and hence messed up the whole question.

15 a) Well done. Most common error involved mixing up +/− signs.

15b) For such simple questions, 15bi and 15bii were poorly done.

15bi) Some students drew a solid line instead of a dashed line.

Curves drawn were untidy and often it was unclear if they were circles, ellipses or some combination of both.

Students are reminded that if they can't draw a neat circle it is advisable to communicate to the examiner your sketch is a circle by

- Plotting 3 points on the circle

or

- stating it is a circle and giving the centre and radius

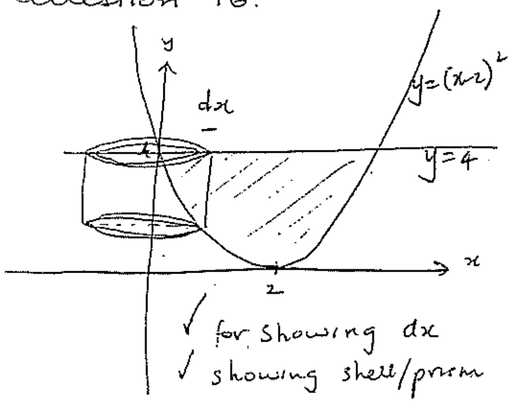
15bii) Many students forgot to circle (1,1). Other common problems were failing to show the angle of $\frac{3\pi}{4}$, drawing a solid rather than dashed line to indicate the beginning of the angle and shading when no shading was required. The arrow head of the ray is also important to demonstrate that it continues forever in that direction.

15 c i) Well done

c ii) Some students skipped second last step showing the common denominator.

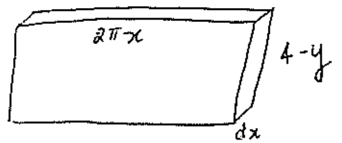
c iii) Many students didn't realise to use 2nd log law and hence messed up the whole question.

Question 16.



✓ for showing dx

✓ showing shell/prism



$$\delta V = 2\pi x\,(4-y)\,dx$$
$$= 2\pi x\,(4-(x-2)^2)\,dx$$
$$= 2\pi x\,(\cancel{4} - x^2 + 4x - \cancel{4})\,dx$$
$$= 2\pi x\,(4x - x^2)$$
$$= 2\pi\,(4x^2 - x^3)$$

$$V = \lim_{\delta x \to 0} \sum_0^4 2\pi x^2\,(4-x)$$

$$V = \int_0^4 2\pi\,(4x^2 - x^3)\,dx$$
$$= 2\pi \int_0^4 4x^2 - x^3\,dx$$
$$= 2\pi \left[\frac{4x^3}{3} - \frac{x^4}{4} \right]_0^4$$
$$= 2\pi \left[\left(\frac{256}{3} - \frac{256}{4} \right) - 0 \right]$$
$$= 2\pi \left(\frac{64}{3} \right)$$
$$= \frac{128}{3}\pi \; u^3$$

(b) i) let a be a multiple roo[t]
$P(a) = 0$

$$P(x) = (x-a)^2\,Q(x)$$

$$P'(x) = 2(x-a)'Q(x) + (x-a)Q'(x)$$

$$P'(a) = 2(a-a)\,Q(a) + (a-a)\,Q'(a)$$
$$= 0.$$

ii)
$$P(x) = 16x^3 - 12x^2 + 1$$
$$P'(x) = 48x^2 - 24x$$
$$= 24x\,(2x-1)$$

$$P'(x) = 0$$
$$x = 0, \quad x = \frac{1}{2}$$

$$P(0) \neq 0$$
$$P\left(\tfrac{1}{2}\right) = 0$$

$\therefore$ $x = \frac{1}{2}$ is the multiple root

$$\left(x - \tfrac{1}{2}\right)^2 = x^2 - x + \tfrac{1}{4}$$

$$
\begin{array}{r|l}
 & 16x + 4 \\
x^2 - x + \frac{1}{4} & 16x^3 - 12x^2 \quad + 1 \\
 & -(16x^3 - 16x^2 + 4x) \\
 & 4x^2 - 4x + 1 \\
 & -(4x^2 - 4x + 1) \\
 & 0
\end{array}
$$

$$P(x) = \left(x - \tfrac{1}{2}\right)^2 (16x + 4)$$

$\therefore$ the roots are
$x = \frac{1}{2}, \frac{1}{2}, -\frac{1}{4}$

(c) i) $\ddot{x} = -4x - 2\sqrt{3}\,\dot{x}$

$$x = Ae^{-\sqrt{3}t}\sin t$$
$$\dot{x} = A\left(-\sqrt{3}\,e^{-\sqrt{3}t}\sin t + e^{-\sqrt{3}t}\cos t\right)$$
$$\dot{x} = Ae^{-\sqrt{3}t}\left(\cos t - \sqrt{3}\sin t\right)$$

$$\ddot{x} = A \cdot -\sqrt{3}\,e^{-\sqrt{3}t}(\cos t - \sqrt{3}\sin t) + Ae^{-\sqrt{3}t}(-\sin t - \sqrt{3}\cos t)$$
$$\ddot{x} = Ae^{-\sqrt{3}t}(-\sqrt{3}\cos t + 3\sin t - \sin t - \sqrt{3}\cos t)$$
$$= Ae^{-\sqrt{3}t}(-2\sqrt{3}\cos t + 2\sin t)$$

$$\ddot{x} = -4x - 2\sqrt{3}\,\dot{x}$$
$$= -4(Ae^{-\sqrt{3}t}\sin t) - 2\sqrt{3}\left(Ae^{-\sqrt{3}t}(\cos t - \sqrt{3}\sin t)\right)$$
$$= -4Ae^{-\sqrt{3}t}\sin t - 2\sqrt{3}\,Ae^{-\sqrt{3}t}(\cos t - \sqrt{3}\sin t)$$
$$= Ae^{-\sqrt{3}t}(-4\sin t - 2\sqrt{3}\cos t + 6\sin t)$$
$$= Ae^{-\sqrt{3}t}(2\sin t - 2\sqrt{3}\cos t)$$

when $t = 0$ $\quad \dot{x} = 3$

$$3 = \dot{x} = 4e^{-\sqrt{3}(0)}(-\sqrt{3}\sin(0) + \cos(0))$$

$$3 = A(1)$$
$$A = 3$$

ii)
$$x = 3e^{-\sqrt{3}t}\sin t$$
$$\dot{x} = 3e^{-\sqrt{3}t}(\cos t - \sqrt{3}\sin t)$$
$$\ddot{x} = 6e^{-\sqrt{3}t}(\sin t - \sqrt{3}\cos t)$$

to find max/min points
$\dot{x} = 0$
$$3e^{-\sqrt{3}t}(\cos t - \sqrt{3}\sin t) = 0$$
$$\sqrt{3}\sin t = \cos t$$

$\tan t = \frac{1}{\sqrt{3}}$

$t = \frac{\pi}{6}, \frac{7\pi}{6}, \frac{13\pi}{6}, \ldots$ ✓

at $t = \frac{\pi}{6}$ $\quad \ddot{x} = 6e^{-\sqrt{3}\cdot\frac{\pi}{6}}\left(\sin\frac{\pi}{6} - \sqrt{3}\cos\frac{\pi}{6}\right)$

$\ddot{x} = -6e^{-\sqrt{3}\frac{\pi}{6}} < 0$

so max at $t = \frac{\pi}{6}$

at $t = \frac{7\pi}{6}$ $\quad \ddot{x} = 6e^{-\sqrt{3}\cdot\frac{7\pi}{6}}\left(\sin\frac{7\pi}{6} - \sqrt{3}\cos\frac{7\pi}{6}\right)$

$= 6e^{-\sqrt{3}\,7\pi/6}$

> 0

so min at $t = \frac{7\pi}{6}$

$\therefore$ travelling down $\quad \frac{\pi}{6} < t < \frac{7\pi}{6}$ ✓

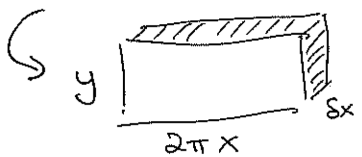
QUESTION 16

(A) (i) Some students did not correctly identify the area to be rotated. Take care!!

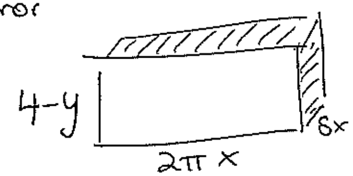
(ii) * Some students were confused to the method of cylindrical shells; incorrectly taking the shell perpendicular to the axis of rotation rather than parallel.

* Many students were unable to find δV correctly. Problems stemmed from poor diagrams of the shell.

* The most common error



INCORRECT



CORRECT.

(B) (i) Mostly well done

Errors included
- incorrect differentiation of $P(x)$
- not explicitly showing why $P'(a) = 0$

(ii) Mostly well done.

(C) (i) Poorly set out.

Consider leaving spaces between lines.

Many silly errors.

(ii) Poorly answered.

* Many students unsure how to approach
* Many students incorrectly assumed $\ddot{x} < 0$ (Not ~~true~~ always true for downwards motion think (over time) this upwards part of the motion has $\ddot{x} < 0$)
* Many students experienced difficulty when trying to solve $\dot{x} < 0$.
* Some students stated two instantaneous times rather than a range.
* Some students used the auxillary angle method which, while still correct, was a longer method for solving.